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Printed in the United States of America

Advanced Science Letters
Vol. 24, No. 3, 1922-1926, 2018

Half-Sweep AOR Iteration With Rotated Nonlocal Arithmetic Mean Scheme For The Solution Of 2D Nonlinear Elliptic Problems

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In this paper, we deal with the application of Half-Sweep Accelerated Over Relaxation (HSAOR) method with nonlocal discretization scheme for solving two-dimensional nonlinear elliptic boundary value problems. To do this, we propose a new nonlocal arithmetic mean scheme namely the four-point rotated nonlocal arithmetic mean scheme being imposed into any nonlinear term in the proposed problems. By using the second order finite difference scheme, the half-sweep nonlinear approximation equation has been derived. Then, the nonlocal discretization scheme is applied to transform the system of nonlinear approximation equations into the corresponding system of linear equations. Throughout numerical results, it can be pointed out that the proposed HSAOR method was superior in terms of number of iterations, execution time and maximum error compared to Full-Sweep Successive Over-relaxation (FSSOR) and Half-Sweep Successive Over Relaxation (HSSOR).

Keywords: Nonlinear Elliptic Boundary Value Problems; Nonlocal Arithmetic Mean Sceheme; HSAOR Iteration

1. INTRODUCTION

Presently, nonlinear boundary value problems of partial differential equations play an essential role in many fields. For instance, numerous nonlinear elliptic boundary value problems can be found in real time application such as numerical weather forecasting, radioactive transfer, optimal control and other areas of physics and engineering. Due to many applications of these problems, obtaining accurate and fast numerical solutions of two-dimensional nonlinear elliptic partial differential equations is of great importance due to its wide application in scientific and engineering researches.

Therefore, many numerical methods intensively have been proposed to solve solve two-dimensional nonlinear elliptic partial differential equations such as non-polynomial spline scheme[1], Pade' approximation[2], collocation method[3], spline scheme[4], finite element methods[5], finite difference methods[6] and numerical integration method[7].

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To get approximate solution of nonlinear elliptic boundary value problems, the paper has considered the finite difference method in order to discretize the proposed nonlinear problems for constructing a corresponding nonlinear system. Then to solve this nonlinear system numerically, this paper has proposed the development of a fast and reliable algorithm to get its approximate solution. According to the previous studies of the case of linear boundary value problems, it can be observed that many researchers have also recommended and derived high-order finite difference approximation equations[8,9,10,11,12]. However in this paper, second order nonlinear half-sweep finite difference approximation equations are used to construct the system of nonlinear equations. Instead of using the Newton method to get numerical solutions of nonlinear system iteratively, the four-point rotated nonlocal arithmetic mean scheme has been applied to the nonlinear system in order to develop the corresponding system of linear equations.

Having any large and sparse linear systems, the iterative methods are apparently the best alternative linear

doi:10.1166/asl.2018.11189

solver that can be considered. As a result, there exist various iterative methods which are used to accelerate convergence rate in solving any linear systems. For instance, in year 1985 Evan[13] has proposed block point iterative methods mainly on the Group Explicit (EG) iterative method. Even though the EG iterative methods are better than the classical point iterative methods, Abdullah has modified the EG method by adding the half-sweep iteration concept and produced Explicit Decoupled Group (EDG) method[14] for solving linear elliptic boundary value problems. Attributes to the advantages of the the half-sweep iteration concept, this concept was extended by many researches[15,16,17,18,19]. As a matter of fact, the main characteristic of half-sweep iteration concept is actually to reduce the computational complexities during iteration process. This is because, the implementation of the half-sweep iterations only consider nearly half of whole node points in a solution domain respectively.

Due to the large scale of the generated sparse linear system, the paper deals with the application of half-sweep iteration concept which is known as Half-sweep AOR method together with the four-point rotated nonlocal arithmetic mean scheme for solving the two-dimensional nonlinear elliptic problems. Hence, the outlines of this paper were organized in following ways: Section 2 will show the formulation of nonlocal discretization approximation scheme. Next, the explanation of FSSOR, HSSOR and HSAOR iterative methods in Section 3 will be given and some numerical results will be shown in Section 4 to state the effectiveness of the proposed methods. Furthermore, the discussion and conclusion are mentioned in Section 5.

Now let us consider a two-dimensional nonlinear elliptic boundary value problem, which is defined in general form as

$$u_{xx} + u_{yy} = F(x, y, u, u_x, u_y) \quad (x, y) \in \Omega = [a,b] \times [a,b] \quad (1)$$

with

$$F(x, y, u, u_x, u_y) = -uu_x - uu_y + f(x, y, u)$$

subject to the boundary conditions

$$u(x, y) = g(x, y), \qquad \partial\Omega$$

where Ω is an arbitrary simply connected bounded region with smooth boundary $\partial\Omega$ and $f(x, y, u)$ and $g(x, y)$ are continuous functions in the respective domain. From Eq.(1), it can be observed that the function $F(x, y, u, u_x, u_y)$ is classified as nonlinear terms.

In formulating the FSSOR, HSSOR and HSAOR iterative methods, the finite grid network needs to be built as a guide for development and implementation of these proposed iterative methods as depicted in Figure 1. According to the figure, let us consider the finite rectangular grid network with spacing grid h and also assume that the spacing grid h in which both directions with $x_i = x_0 + ih,\ y_j = y_0 + jh$ are defined as

$$h = \Delta x = \Delta y = \frac{(b-a)}{m} \quad (2)$$

where Δx and Δy represent the increment of x and y directions respectively while m is number of subintervals. Then let $U(x_i, y_i) = u_{i,j}$ indicates the approximation value of function u at point (x_i, y_i).

2. FORMULATION OF ROTATED NONLOCAL DISCRETIZATION SCHEME

Before presenting the derivation of the half-sweep nonlinear finite difference approximation equation of Problem (1), let us consider several four-point standard nonlocal arithmetic mean discretization schemes be given as follows[20]

$$U_{ij}^2 = \left(\frac{U_{i-1,j} + U_{i+1,j} + U_{i,j-1} + U_{i,j+1}}{4} \right) U_{i,j} \quad (3)$$

$$U_{ij}^3 = \left(\frac{U_{i-1,j} + U_{i+1,j} + U_{i,j-1} + U_{i,j+1}}{4} \right) U^2{}_{i,j} \quad (4)$$

In this study, the formulation in Eq.(4) will be adopted to derive the four-point rotated nonlocal arithmetic mean discretization scheme for two dimensional problem given as

$$U_{i,j}^2 = \left(\frac{U_{i-1,j-1} + U_{i+1,j+1} + U_{i-1,j+1} + U_{i+1,j-1}}{4} \right) U_{i,j} \quad (5)$$

Using the approach of second-order finite difference discretization scheme, the half-sweep second-order finite difference approximation equations for problem (1) can be easily shown as

$$\begin{aligned} &U_{i-1,j-1} + U_{i+1,j+1} + U_{i-1,j+1} + U_{i+1,j-1} \\ &- 4U_{i,j} = 2h^2 F_{i,j} \end{aligned} \quad (6)$$

where

$$F_{i,j} = F\left(x_i, y_j, U_{ij}, \frac{U_{i+1,j+1} - U_{i-1,j-1}}{2\left(\sqrt{2h}\right)}, \frac{U_{i-1,j+1} - U_{i+1,j-1}}{2\left(\sqrt{2h}\right)} \right).$$

Then, by simplifying the Eq. (7) can be shown as

$$\begin{aligned} &U_{i-1,j-1} + U_{i+1,j+1} + U_{i-1,j+1} + U_{i+1,j-1} \\ &- 4U_{i,j} - 2h^2 F_{ij} = 0, \end{aligned} \quad (7)$$

where

$$F_{ij} = g\left(\begin{matrix} x_i, y_j, \dfrac{U_{i-1,j-1} + U_{i+1,j+1} + U_{i-1,j+1} + U_{i+1,j-1}}{4}, \\ \dfrac{U_{i+1,j+1} - U_{i-1,j-1}}{2\left(\sqrt{2h}\right)}, \dfrac{U_{i-1,j+1} - U_{i+1,j-1}}{2\left(\sqrt{2h}\right)} \end{matrix} \right) \quad (8)$$

Actually, Eq. (9) is called as the nonlinear term of the problem (1). To solve the nonlinear system in Eq. (8), the nonlocal discretization scheme is used to transform the nonlinear system into the form of a system of linear equations. In this paper, however, we consider the nonlocal discretization scheme in Eq. (4) being imposed over the nonlinear approximation equation (10). Therefore, Eq. (9) can be rewritten as follows

$$F_{i,j} = 2h^2 F\left(\begin{array}{l} x_i, y_j, \dfrac{U_{i-1,j-1} + U_{i+1,j+1} + U_{i-1,j+1} + U_{i+1,j-1}}{4}, \\ \dfrac{u_{i+1,j+1} - u_{i-1,j-1}}{2\left(\sqrt{2h}\right)}, \dfrac{u_{i-1,j+1} - u_{i+1,j-1}}{2\left(\sqrt{2h}\right)} \end{array} \right) \quad (9)$$

To solve the nonlinear equation (8) iteratively, the four-point rotated nonlocal arithmetic mean discretization scheme is used to transform the nonlinear equation into a linear equation. In this paper, however, we consider the nonlocal discretization scheme in Eq.(5) being imposed over the nonlinear approximation equation (7).

3. FORMULATION OF HSAOR ITERATIVE METHOD

Let us consider the nonlinear equation in Eq.(8) that will be solved by using HSAOR, HSSOR and FSSOR iterative methods. In this section, we start on how to derive the formulation of the SOR iteration family. Therefore, based on the half-sweep approximation equation in Eq. (7) the general scheme of the FSSOR method can be stated as[21,22,23]

$$U_{i,j}^{(k+1)} = \frac{\omega}{4}\left(\begin{array}{l} U_{i-1,j}^{(k+1)} + U_{i+1,j}^{(k)} + U_{i,j-1}^{(k+1)} + U_{i,j+1}^{(k)} \\ -h^2 f_{ij} \end{array} \right) + (1-\omega)U_{i,j}^{(k)} \quad (10)$$

where ω and $U_i^{(k)}$ represent as a relaxation factor and the kth represent as a relaxation factor and the kth the estimation for corresponding exact solutions respectively. Actually the FSSOR iterative method in Eq. (12) was proposed by Young[24,25,26]. Apart from this, Abdullah[14] first introduces the concept of the half-sweep iteration in solving any system of linear equations. The concept of half-sweep iteration is actually to reduce the computational complexities during iteration process. Thus, in this paper, we implement the HSSOR iterative method in solving the nonlinear elliptic boundary value problem. Based on Eq. (8) the general scheme for HSSOR iterative method can be given as

$$U_{i,j}^{(k+1)} = \frac{\omega}{4}\left(\begin{array}{l} U_{i-1,j-1}^{(k+1)} + U_{i+1,j+1}^{(k)} + U_{i-1,j+1}^{(k)} \\ + U_{i+1,j-1}^{(k+1)} - 2h^2 f_{ij} \end{array} \right) + (1-\omega)U_{i,j}^{(k)} \quad (11)$$

for $j = 1, 2, 3 \cdots, n-1$ and $i = 2-(j\%2), i \le n-1, i++$. Due to the advantage of the SOR iteration family with one parameter ω, Hadjidimos[24] has proposed the Accelerated Over Relaxation (AOR) iterative method which involves two-parameters, r and ω. However, this AOR method can be categorized as one of the full-sweep iterative methods. Basically these two arbitrary parameters can be fully exploited to produce iterative methods that have faster rates of convergence. Thus, in this paper, we investigate the effectiveness of the HSAOR iterative method in solving two-dimensional nonlinear elliptic boundary value problems in Eq. (1). Based on Eq. (11), the general scheme for the HSAOR method can be given as

$$\begin{aligned} U_{i,j}^{(k+1)} &= \frac{\omega}{4}\left(U_{i-1,j-1}^{(k)} + U_{i+1,j+1}^{(k)} + U_{i-1,j+1}^{(k)} + U_{i+1,j-1}^{(k)} - 2h^2 f_{ij} \right) \\ &+ \frac{r}{4}\left(\left(U_{i-1,j-1}^{(k+1)} - U_{i-1,j-1}^{(k)} \right) + \left(U_{i+1,j-1}^{(k+1)} - U_{i+1,j-1}^{(k)} \right) \right) \\ &+ (1-\omega)U_{i,j}^{(k)} \end{aligned} \quad (12)$$

for $j = 1, 2, 3 \cdots, n-1$ and $i = 2-(j\%2), i \le n-1, i++$.

As taking $\omega = r$, the HSAOR method reduces to the HSSOR method, whereas choosing $\omega = r = 1$, this method is called as the Half-Sweep Gauss-Seidel (HSGS) iterative method. In this study, however, the FSSOR iterative methods will be used as a control method. The general algorithm for the HSAOR iterative methods to solve the linear equation (7) would be generally described in Algorithm 1. For the implementation of the AOR iteration family, a good choice for the value of the parameters r and ω can be used to accelerate the convergence rate of the iteration process. In practice, the optimal value of both parameters r and ω will be obtained by implementing several computer programs and then the best approximate value of these parameters is chosen in which its number of iterations is the smallest. Then the general algorithm for the HSAOR iterative method in Eq.(12) would be described in Algorithm 1.

Algorithm 1 : HSAOR Scheme

i. Initialize $\underset{\sim}{U}^{(0)} \leftarrow 0$ and $\varepsilon \leftarrow 10^{-10}$

ii. Assign the optimal value of r and ω

iii. For $j = 1, 2, 3 \cdots, n-1$ and $i = 2-(j\%2), i \le n-1, i++$

a. Calculate

$$F_{i,j} = F\left(\begin{array}{l} x_i, y_i, U_{i,j}, \dfrac{U_{i+1,j+1} - U_{i-1,j-1}}{2\left(\sqrt{2h}\right)}, \\ \dfrac{U_{i-1,j+1} - U_{i+1,j-1}}{2\left(\sqrt{2h}\right)} \end{array} \right)$$

b. Calculate

$$\begin{aligned} U_{i,j}^{(k+1)} &= \frac{\omega}{4}\left(\begin{array}{l} U_{i-1,j-1}^{(k)} + U_{i+1,j+1}^{(k)} + U_{i-1,j+1}^{(k)} \\ + U_{i+1,j-1}^{(k)} - 2h^2 f_{ij} \end{array} \right) \\ &+ \frac{r}{4}\left(\left(U_{i-1,j-1}^{(k+1)} - U_{i-1,j-1}^{(k)} \right) + \left(U_{i+1,j-1}^{(k+1)} - U_{i+1,j-1}^{(k)} \right) \right) \\ &+ (1-\omega)U_{i,j}^{(k)} \end{aligned}$$

iv. Convergence test. If the convergence criterion i.e $\left\| \underset{\sim}{U}^{(k+1)} - \underset{\sim}{U}^{(k)} \right\| \le \varepsilon = 10^{-10}$ is satisfied, go step (v). Otherwise go back to step (iii)

v. Display approximate solutions

4. NUMERICAL EXPERIMENTS

In order to validate the performance of the FSSOR, HSSOR, and HSAOR iterative methods together with the four-point rotated nonlocal arithmetic mean scheme, three nonlinear elliptic example problems were tested. For the sake of comparison, three criteria will be considered for these three proposed iterative methods which are number of iterations, execution time (in seconds) and maximum absolute error.

Example 1[25]

$$u_{xx} + u_{yy} + uu_x + uu_y = \exp(u), \quad (x, y) \in \Omega = [0,1] \times [0,1] \quad (13)$$

the exact solution of problem (13) was defined by

$$u(x, y) = x^2 + y, \quad \partial\Omega$$

Example 2[25]

$$u_{xx} + u_{yy} + uu_x + uu_y = \exp(u), \quad (x, y) \in \Omega = [0,1] \times [0,1] \quad (14)$$

And the exact solution of problem (14) was defined by

$$u(x, y) = \exp(-x)\sin \pi y, \quad \partial\Omega$$

Example 3[25]

$$u_{xx} + u_{yy} - uu_y = \exp(-x)\sin \pi y\left(1 - \pi^2 - \pi \exp-(x)\cos \pi y\right), (x, y) \in \Omega = [0,1] \times [0,1] \quad (15)$$

And the exact solution of problem (15) was defined by

$$u(x, y) = \exp(-x)\sin \pi y, \quad \partial\Omega$$

Following to these three examples, all the results of numerical experiments obtained by the FSSOR, HSSOR, and HSAOR iterative methods have been recorded in Table 1.

Clearly it can be observed from Table 1 that the FSSOR, HSSOR, and HSAOR have been implemented at different grid sizes 32, 64, 128, 256, and 512. According to numerical results in Table 1 for the case of second order finite difference schemes, the HSAOR iterative method with the four-point rotated nonlocal arithmetic mean scheme shows the superiority in terms of number of iterations and the execution time compared with the FSSOR and HSSOR methods.

5. CONCLUSION

In this research, we present the performance of the HSAOR iterative method associated with the four-point rotated nonlocal arithmetic mean scheme and the second order finite difference approximation scheme for solving the nonlinear elliptic boundary value problems. Based on Table 1, numerical results showed that HSAOR method solved the proposed problems with least number of iterations as compared to the HSSOR and FSSOR methods. Meanwhile, in terms of execution time, HSAOR method computes with the fastest time for all considered mesh sizes. In the aspect of accuracy, numerical solutions obtained for test problems 1 to 3 are comparable for all the tested iterative methods. Finally, it can be concluded that the HSAOR method is superior to HSSOR and FSSOR methods. This is mainly because of the implementation of the HSAOR method with parameters r and ω that can be used to accelerate the convergence rate of the iteration process. For future works, this study can be extended to investigate the performance analysis of block point iterative methods[26,27] and weighted parameter iterative methods[28,29].

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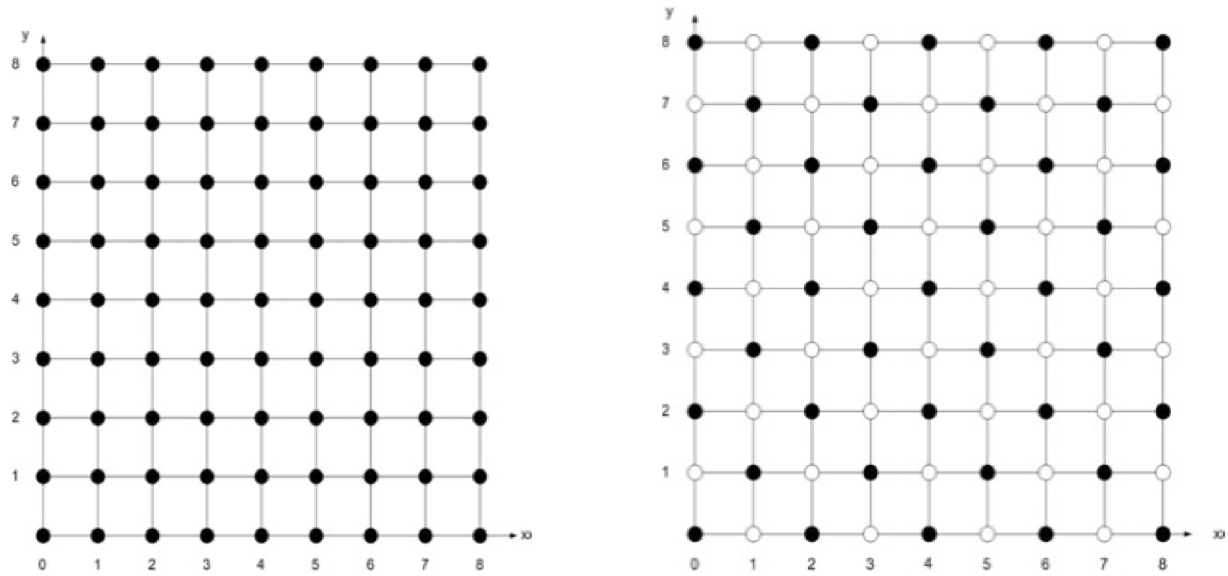

(a) **(b)**

Figure.1. Finite grid networks for the full-sweep (a) and half-sweep (b) in case m=8

TABLE 1. Comparison between number of iterations (K), the execution time (seconds) and maximum errors for the iterative methods using example at grid sizes 32,64,128,256,512

	EXAMPLE 1			EXAMPLE 2			EXAMPLE 3		
Number of iterations K									
M	**FSSOR**	**HSSOR**	**HSAOR**	**FSSOR**	**HSSOR**	**HSAOR**	**FSSOR**	**HSSOR**	**HSAOR**
32	136	97	92	135	91	87	130	90	86
64	266	194	181	258	184	171	259	182	168
128	524	387	356	522	365	341	514	358	334
256	1035	772	728	1026	725	664	1025	713	673
512	2058	1538	1487	2050	1439	1264	2049	1416	1293
Execution time (seconds)									
M	**FSSOR**	**HSSOR**	**HSAOR**	**FSSOR**	**HSSOR**	**HSAOR**	**FSSOR**	**HSSOR**	**HSAOR**
32	0.11	0.06	0.03	0.11	0.05	0.02	0.11	0.05	0.02
64	0.39	0.33	0.13	0.34	0.12	0.09	0.34	0.26	0.13
128	2.73	1.21	0.79	1.95	1.23	0.55	1.30	0.70	0.47
256	21.27	9.85	6.02	8.12	6.80	4.60	8.67	3.83	3.31
512	168.81	67.81	51.28	62.80	48.26	33.21	65.14	32.19	28.06
Maximum errors									
M	**FSSOR**	**HSSOR**	**HSAOR**	**FSSOR**	**HSSOR**	**HSAOR**	**FSSOR**	**HSSOR**	**HSAOR**
32	9.62e-02	5.89e-02	5.70e-02	2.87e-04	2.41e-02	2.41e-02	3.24e-04	1.80e-02	1.80e-02
64	9.63e-02	5.87e-02	5.71e-02	7.17e-05	2.41e-02	2.42e-02	8.12e-05	1.81e-02	1.81e-02
128	9.63e-02	5.87e-02	5.80e-02	1.79e-05	2.42e-02	2.42e-02	2.03e-05	1.82e-02	1.82e-02
256	9.63e-02	5.87e-02	5.85e-02	4.48e-06	2.42e-02	2.42e-02	5.07e-06	1.82e-02	1.82e-02
512	9.63e-02	5.87e-02	5.86e-02	1.12e-06	2.42e-02	2.42e-02	1.27e-06	1.82e-02	1.82e-02