Jurnal KALAM *Jurnal Karya Asli Lorekan Ahli Matematik Vol. 8 No.1 (2015) Page 016-022*

Jurnal Karya Asli Lorekan Ahli Matematik

HALF-SWEEP GAUSS-SEIDEL ITERATION APPLIED TO TIME-FRACTIONAL DIFFUSION EQUATIONS

A. Sunarto[1], J. Sulaiman[2] and A. Saudi[3]

[1,2] *School of Science and Technology, Universiti Malaysia Sabah,88400 Kota Kinabalu, Sabah, Malaysia*

[3]*School of Engineering and Information Technology, Universiti Malaysia Sabah, 88400 Kota Kinabalu, Sabah, Malaysia*

[1]andang99@gmail.com, [2]jumat@ums.edu.my, [3]azali60@gmail.com

Abstract : In this study, we derive a finite difference approximation equation from the discretization of the one-dimensional linear time-fractional diffusion equations by using the Caputo's time fractional derivative. A linear system will be generated by the Caputo's finite difference approximation equation. Then the resulting of the linear system has been solved using Half-Sweep Gauss-Seidel (HSGS) iterative method in which its effectiveness will be compared with the existing Gauss-Seidel method (known as Full-Sweep Gauss-Seidel (FSGS)). An example of the problem is presented to test the effectiveness the proposed method. The findings of this study show that the proposed iterative method is superior compared with the FSGS method.

Keywords: Caputo's fractional derivative; Implicit scheme; HSGS method.

1.0 Introduction

According to previous studies [1,2,3,4,5] the use of fractional partial differential equations (FPDEs) have attracted many researchers in mathematics, physics, engineering and chemistry to obtain a numerical and/or analytical solutions of the fractional problems. For Instant, a fractional derivative replaces the first-order space partial derivative in a diffusion model and lead to slower diffusion [5]. So to solve a one-dimensional diffusion model with constant coefficients, analytical solutions are available using iterative methods.

Based on numerical techniques applied to the time fractional diffusion equations (TFDE), many proposed methods have been initiated such as transform methods [6], finite elements together with the method of lines [3], explicit and implicit finite difference methods [3,7]. Although the explicit methods are conditionally stable, this finite difference schemes are available in the literature [8].

Implicit study, The time fractional diffusion equations (TFDE) problem will be discretised.By imposing the implicit finite difference scheme and Caputo fractional operator the approximation equations can be used to construct a linear system at each time level. To solve the linear systems, the concept of the iterative methods have been discussed by many researchers such as Young [9], Hackbusch [10] and Saad [11] and it can be observed that there are several families of iterative methods. In addition to these iterative methods, the concept of block iteration has also been introduced by Evans [13], Ibrahim and Abdullah [12], Yousif and Evans [14,23] to show the efficiency of its computation cost.

The main objective of this paper is to study the effectiveness of the Half-Sweep Gauss-Seidel (HSGS) iterative method for solving time-fractional parabolic partial differential equations (TPPDE's) by using the Caputo's implicit finite difference approximation equation. To show the capability of the HSGS method, we also implement the Full-Sweep Gauss Seidel (FSGS) iterative methods being used as a control method.

To indicate the efficiency of this HSGS method, let us consider time-fractional diffusion equation (TFDE's) be defined as

$$\frac{\partial^{\alpha}U(x,t)}{\partial^{\alpha}} = a(x)\frac{\partial^{2}U(x,t)}{\partial x^{2}} + b(x)\frac{\partial U(x,t)}{\partial x} + c(x)U(x,t) \tag{1}$$

Published by Pustaka Aman Press Sdn. Bhd.

where *a(x), b(x)* and *c(x)* are known functions or constants, whereas α is a parameter which refers to the fractional order of time derivative.

The organization of the paper is as follows: In Section 2 and 3, we approximate the formula of the Caputo's fractional derivative operator and numerical procedure for solving time fractional diffusion equation (1) by means of the implicit finite difference method. In Section 4, formulation of the HSGS iterative method is introduced. In Section 5 we give a numerical example and the results and conclusion are given in Section 6.

2. 0 Preliminaries

Before developing the discrete equation of Problem (1). We introduce some basic definitions

Definition 2.1 [8] The Riemann-Liouville fractional integral operator , J^{α} of order-α is defined as

$$J^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha} f(t) dt \, , \alpha > 0 \, , x > 0 \tag{2}$$

Definition 2.2[8] The Caputo's fractional partial derivative operator, D^{α} of order -α is defined as

$$D^{\alpha} f(x) = \frac{1}{\Gamma(m-\alpha)} \int_0^x \frac{f^{(m)}(t)}{(x-t)^{\alpha-m+1}} dt, \quad \alpha > 0 \tag{3}$$

$$\text{with } m-1 < \alpha \leq m, \mathrm{m} \in \mathrm{N}, \ x > 0$$

The purpose of this paper is examine the Half-Sweep Gauss-Seidel (HSGS) iterative method which is compared with the Full-Sweep Gauss-Seidel (FSGS) iterative method for solving Problem (1) with variable coefficients. In solving numerical of Problem (1), we derive numerical approximations based on the Caputo's derivative definition with Dirichlet boundary conditions and consider the non-local fractional derivative operator. This approximation equation can be categorized as unconditionally stable scheme. On strength of Problem (1), the solution domain of the problem has been restricted to the finite space domain $0 \leq x \leq \alpha$, with $0 < \alpha < 1$, whereas the parameter α refers to the fractional order of space derivative. To solve of Problem (1), let us consider the initial boundary conditions of Problem (1)

$$U(0,t) = g_0(t), \text{ and t } \ U(x,0) = f(x),$$

where $g_0(t), g_1(t),$ and $f(x),$ are given functions. For a discretize approximation to the time fractional derivative in Eq. (1), we consider Caputo's fractional partial derivative of order α , defined by [8,9]

$$\frac{\partial^{\alpha} u(x,t)}{\partial t^{\alpha}} = \frac{1}{\Gamma(n-1)} \int_0^{\infty} \frac{\partial u(x-s)}{\partial t} (t-s)^{-\alpha} ds, \quad t > 0, \quad 0 < \alpha < 1 \tag{4}$$

3.0 Caputo's Implicit Finite Difference Approximation

Based on Eq. (4), the formulation of Caputo's fractional partial derivative of the first order approximation method is given as

$$D_t^{\alpha} U_{i,n} \cong \sigma_{\alpha,k} \sum_{j=1}^{n} \omega_j^{(\alpha)} \left(U_{i,n-j+1} - U_{i,n-j} \right) \tag{5}$$

where

$$\sigma_{\alpha,k} = \frac{1}{\Gamma(1-\alpha)(1-\alpha)k^{\alpha}}$$

and

$\omega_j^{(\alpha)} = j^{1-\alpha} - (j-1)^{1-\alpha}$.

Before discretizing Problem (1), we assume that the solution domain of the problem be partitioned uniformly. To do this, we consider some positive integers *m* and *n* in which the grid sizes in space and time directions for the finite difference algorithm are defined as $h = \Delta x = \frac{\gamma - 0}{m}$ and $k = \Delta t = \frac{T}{n}$ respectively. Based on these grid sizes, we construct the uniformly grid network of the solution domain where the grid points in the space interval $[0,\gamma]$ are indicated as the numbers $x_i = ih,\ i = 0,1,2,...,m$ and the grid points in the time interval $[0,T]$ are labeled $t_j = jk,\ j = 0,1,2,...,n$. Then the values of the function $U(x,t)$ at the grid points are denoted as $U_{i,j} = U(x_i, t_j)$. By using Eq. (5) and the implicit finite difference discretization scheme, the Caputo's implicit finite difference approximation equation of Problem (1) to the grid point centered at $(x_i, t_j) = (ih, nk)$ is given as

$$\sigma_{\alpha,k} \sum_{j=1}^{n} \omega_j^{(\alpha)} \left(U_{i,n-j+1} - U_{i,n-j}\right)$$

$$= a_i \frac{1}{4h^2}\left(U_{i-2,n} - 2U_{i,n} + U_{i+2,n}\right) + b_i \frac{1}{4h}\left(U_{i+2,n} - U_{i-2,n}\right) + c_i U_{i,n}, \qquad (6)$$

for *i=1,2...,m-1.*

According to Eq. (6), this approximation equation is known as the fully implicit finite difference approximation equation which is consistent first order accuracy in time and second order in space. Basically, the approximation equation (6) can be rewritten based on the specified time level. For instance, we have for $n \geq 2$:

$$\sigma_{\alpha,k} \sum_{j=1}^{n} \omega_j^{(\alpha)} \left(U_{i,n-j+1} - U_{i,n-j}\right) = \left(\frac{a_i}{4h^2} - \frac{b_i}{4h}\right) U_{i-2,n} + \left(c_i - \frac{a_i}{2h^2}\right) U_{i,n} + \left(\frac{a_i}{4h^2} + \frac{b_i}{4h}\right) U_{i+2,n}, \qquad (7a)$$

$$\therefore \sigma_{\alpha,k} \sum_{j=1}^{n} \omega_j^{(\alpha)} \left(U_{i,n-j+1} - U_{i,n-j}\right) = p_i U_{i-2,n} + q_i U_{i,n} + r_i U_{i+2,n},$$

where

$$p_i = \frac{a_i}{4h^2} - \frac{b_i}{4h},$$

$$q_i = c_i - \frac{a_i}{2h^2},$$

$$r_i = \frac{a_i}{4h^2} + \frac{b_i}{4h}$$

Also for *n = 1,*

$$-p_i U_{i-2,1} + q_i^* U_{i,1} - r_i U_{i+2,1} = f_{i,0}, \quad i = 1,2,...,m-1 \qquad (7b)$$

where

$$\omega_j^{(\alpha)} = 1,$$

$$q_i^* = \sigma_{\alpha,k} - q_i,$$

$$f_{i,0} = \sigma_{\alpha,k} U_{i,0}.$$

According to Eq. (7b), it can be seen that the tridiagonal linear system can be constructed in matrix form as

$$A\underset{\sim}{U} = \underset{\sim}{f} \tag{8}$$

where

$$A = \begin{bmatrix} q_1^* & -r_1 & & & & \\ -p_2 & q_2^* & -r_2 & & & \\ & -p_3 & q_3^* & -r_3 & & \\ & & \ddots & \ddots & \ddots & \\ & & & -p_{m-2} & q_{m-2}^* & -r_{m-2} \\ & & & & -p_{m-1} & q_{m-1}^* \end{bmatrix}_{(m-1)x(m-1)},$$

$$\underset{\sim}{U} = \left[U_{11} \quad U_{21} \quad U_{31} \quad \cdots \quad U_{m-2,1} \quad U_{m-1,1} \right]^T,$$

$$\underset{\sim}{f} = \left[U_{11} + p_1 U_{01} \quad U_{21} \quad U_{31} \quad \cdots \quad U_{m-2,1} \quad U_{m-1,1} + p_{m-1} U_{m,1} \right]^T.$$

4.0 Formulation of Half-Sweep Gauss-Seidel

As mentioned in section 4 and see , the coefficient matrix A of Eq.(8) has a large scale and sparse. The concept of various iterative methods has been initiated and conducted by many researchers such as, Young [9], Hackbusch [10], Saad [11], Evans [13], Yousif and Evans [14,23], and Othman and Abdullah [21]. To Solve the tridiagonal linear system, Abdullah [24] initiated Half-Swep iteration, which is the most known and widely used iterative techniques to solve linear systems.In addition to that, the iteration has been extensively used by many researchers; see Ibrahim and Abdulah [12], Othman and Abdullah [21], Suliaman et al.[20,22], Aruchunan and Sulaiman [17,18,19], Muthuvalu and Sulaiman [16] and Yousif and Evans [14,23].The main advantage of the Half-Sweep iteration is to reduce the computational complexities during iteration process. As a result of this concept and using HSGS method, the coefficient matrix of the linear system (8) can be expressed as summation of the three matrices

$$A = D - L - V \tag{11}$$

where D, L and V are diagonal, lower triangular and upper triangular matrices respectively.
Thus, Half-Sweep Gauss-Seidel iterative method can be defined generally as

$$\underset{\sim}{U}^{(k+1)} = \left(D - L\right)^{-1} \left(V \underset{\sim}{U}^{(k)} + \underset{\sim}{f} \right)$$

(12)

where $\underset{\sim}{U}^{(k)}$ represents an unknown vector at k^th iteration. The implementation of the HSGS iterative method can be described in Algorithm 1.

Algorithm 1: HSGS method

i. Initializing all the parameters. Set $k = 0$.

ii. For $i = p, 2p, \ldots, n-2p, n-p, n$ and $i = p, 2p, \ldots, n-2p, n-p, n$

Calculate

$$\tilde{U}_i^{(k+1)} = \frac{1}{M_{i,i}} \left(f_i - \sum_{j=p,2p}^{i-p} M_{i,j} \tilde{U}_j^{(k+1)} - \sum_{j=i+p,i+2p}^{n} M_{i,j} \tilde{U}_j^{(k)} \right)$$

iii. Convergence test. If . $\left\| \tilde{U}^{(k+1)} - \tilde{U}^{(k)} \right\| \le \varepsilon = 10^{-10}$ is satisfied, go to Step (iv). Otherwise go back to Step (ii).

iv. Display approximate solutions.

5.0 Numerical Experiment

In this section, an example of the time fractional diffusion equation is given to illustrate the accuracy and effectiveness properties of the Full-Sweep Gauss-Seidel (GS) and Half-Sweep Gauss-Seidel (HSGS) iterative methods. For comparison purposes three criteria have been considered such as number of iterations, execution time (in seconds) and maximum absolute error at three different values of $\alpha = 0.25$, 0.50 and 0.75. For implementation of three iterative schemes, the convergence test considered the tolerance error, which is fixed as $\varepsilon = 10^{-10}$.
Let us consider the time fractional initial boundary value problem be given as [15]

$$\frac{\partial^\alpha U(x,t)}{\partial t^\alpha} = \frac{\partial^2 U(x,t)}{\partial x^2}, \quad 0 < \alpha \le 1, 0 \le x \le \gamma, \quad t > 0, \tag{13}$$

where the boundary conditions are stated in fractional terms

$$U(0,t) = \frac{2kt^\alpha}{\Gamma(\alpha+1)}, \; U(\ell,t) = \ell^2 + \frac{2kt^\alpha}{\Gamma(\alpha+1)}, \tag{14}$$

and the initial condition

$$U(x,0) = x^2. \tag{15}$$

From Problem (13), as taking $\alpha = 1$, it can be seen that Eq. (13) can be reduced to the standard diffusion equation

$$\frac{\partial U(x,t)}{\partial t} = \frac{\partial^2 U(x,t)}{\partial x^2}, \quad 0 \le x \le \gamma, \quad t > 0, \tag{16}$$

subjected to the initial condition

$$U(x,0) = x^2,$$

and boundary conditions

$$U(0,t) = 2kt, \qquad U(\ell,t) = \ell^2 + 2kt,$$

Then the analytical solution of Problem (16) is obtained as follows

$$U(x,t) = x^2 + 2kt.$$

Now by applying the series

$$U(x,t) = \sum_{n=0}^{m-1} \frac{\partial^n U(x,0)}{\partial t^n} \frac{t^n}{n!} + \sum_{n=1}^{\infty} \sum_{i=0}^{m-1} \frac{\partial^{mn+i} U(x,0)}{\partial t^{mn+i}} \frac{t^{n\alpha+i}}{\Gamma(n\alpha+i+1)}$$

to $U(x,t)$ for $0 < \alpha \le 1$, it can be shown that the analytical solution of Problem (13) is given as

$$U(x,t) = x^2 + 2k\frac{t^\alpha}{\Gamma(\alpha+1)}.$$

All results of numerical experiments for Problem (13), which were obtained from implementation of FSGS and HSGS iterative methods have been recorded in Table 1 at different values of mesh sizes, *m* = 128, 256, 512, 1024, and 2048.

TABLE 1. Comparison of number iterations, the execution time (seconds) and maximum errors for the iterative methods using example at $\alpha = 0.25, 0.50, 0.75$

M	Method	α = 0.25			α = 0.50			α = 0.75		
		K	Time	Max Error	K	Time	Max Error	K	Time	Max Error
128	**FSGS**	21017	37.01	9.97e-05	13601	23.92	9.85e-05	6695	12.1	1.30e-04
	HSGS	5682	5.28	9.96e-05	3671	3.77	9.84e-05	1805	2.18	1.29e-04
256	**FSGS**	77231	332.11	1.00e-04	50095	213.28	9.90e-05	24732	104.04	1.30e-04
	HSGS	21017	34.81	9.97e-04	13601	22.72	9.85e-05	6695	11.41	1.30e-04
512	**FSGS**	281598	2522.20	1.02e-04	183181	162.08	1.01e-05	90783	831.58	1.32e0-4
	HSGS	77231	333.92	1.00e-04	50095	214.58	9.90e-05	24732	105.44	1.30e-04
1024	**FSGS**	1017140	18485.43	1.09e-04	663971	2454.53	1.08e-04	330622	5870.9	1.40e-04
	HSGS	90783	771.42	1.32e-04	183181	1568.23	1.00e-04	90783	771.42	1.32e-04
2048	**FSGS**	3631638	58914.30	1.38e-04	2380946	17795.25	1.38e-04	1192528	8794.26	1.71e-04
	HSGS	1017140	17798.81	1.09e-04	663971	11482.81	1.4e-04	330622	5653.5	1.40e-04

6.0 Conclusion

For the numerical solution of the time fractional diffusion problems, the paper presents the derivation of the Caputo's implicit finite difference approximation equations in which this approximation equation leads to a tridiagonal linear system. From observation of all experimental results by imposing the FSGS and HSGS iterative methods, it is obvious at $\alpha = 0.25$ that number of iterations have declined approximately by 71.99-91.07% corresponds to the HSGS iterative method compared with the FSGS methods. Again in terms of execution time, implementations of HSGS method are much faster about 69.78-95.82% than the FSGS methods. It means that the HSGS method requires smaller number of iterations and computational time at $\alpha = 0.25$ as compared with FSGS iterative methods. Based on the accuracy of both iterative methods, it can be concluded that their numerical solutions are in good agreement.

References

[1] Mainardi, F, 1997, Fractals and Fractional Calculus Continuum Mechanics. Heidelberg: Springer-Verlag, 291-348.

[2] Diethelm, K. and Freed, A.D, 1999, On the solution of nonlinear fractional order differential equation used in the modeling of viscoelasticity, in Scientific Computing in Chemical engineering II Computational Fluid Dynamic, Reaction Engineering and Molecular Properties, Heidelberg: Springer-Verlag, 217-224.

[3] Meerschaert, M.M. and Tadjeran, C, 2004, Finite difference approximation for fractional advection-dispersion flow equations, *JCAM*, 172(2004), 145-155.

[4] Chaves, A, 1998, Fractional diffusion equation to describe Levy flight, *Phys.Lett, A*239, 13-16.

[5] Agrawal, O.P, 2002, Solution for a fractional diffusion-wave equation defined in a bounded domain, *Nonlinear Dynamic*, 29(2002)**,** 145-155

[6] Yuste, S.B. andAcedo, L, 2005, An explicit finite difference method and a new von Neumann-type stability analysis for fractional diffusion equations, *SIAM J. Numer. Anal*.42 (5) (2005), 1862-1874

[7] Yuste, S.B, 2006, Weighted average finite difference method for fractional diffusion equations, *J.Comp.Phys*.216 (2006),264-274

[8] Zhang, Y, 2009, A Finite Difference Method For Fractional Partial Differential Equation, *Applied Mathematics And Computation 215* (2009),524-529.

[9] Young, D.M, 1971, Iterative solution of large linear systems. London: Academic Press.

[10] Hackbusch, W, 1995, Iterative solution of large sparse systems of equations. New York: Springer-Verlag.

[11] Saad, Y, 1996, Iterative method for sparse linear systems. Boston: International Thomas Publishing.

[12] Ibrahim, A and Abdullah, A.R, 1995, Solving the two dimensional diffusion equation by the Four Point Explicit Decoupled Group (EDG) iterative method. *International Journal Computer Mathematics*. 58 (1995),253-256.

[13] D.J. Evans. 1985, Group Explicit Iterative methods for solving large linear systems, *Int. J. Computer Maths.,* **17**:81-108.

[14] Yousif, W. S and Evans, D.J, Explicit Group Iterative Methods for solving elliptic partial differential equations in 3-space dimensions. *Int. J. Computer Maths.,*18 (1986)**,** 323-340.

[15] Ali, S E, Ozgur, B and Korkmaz, E, Analysis Of Fractional Partial Differential Equations By Taylor Series Expansion. Boundary Value Problem, A Springer Open, (2013),68.

[16] Muthuvalu, M.S and Sulaiman, J, Computational Methods Based on Complexity Reduction Approach for First ind Linear Fredholm Integral Equations with Semi-Smooth Kernel, *Jurnal Of Modern in Numeric Mathematics*, (2012), 64-71.

[17] Aruchunan, E and Sulaiman, J, Half-Sweep Conjugate Gardient method for Solving First Order Linear Fredholm Integro-differential Equations, *Australian Journal of Basic and applied Sciences*, (2011) 38-43.

[18] Aruchunan, E and Sulaman, J. Half –Sweep Quadrature-Difference Schemes with Iterative Method In Solving Linear Fredholm Integro-Differential Equations, *Progress in Applied Mathematics*, (2013) 11-21.

[19] Aruchunan, E and Sulaiman, J. Applications of the Central-Difference with Half-Sweep Gauss-Seidel Method for Solving First Order Linear Fredholm Integro-Differential Equations, *World Academy of Science, Engineering and Technology*, (2012), 335-339.

[20] Sulaiman, J, Othman, M and Hasan, M.K, A New Half-Sweep Algebraic Multigrid (HSAMG) Algorithm for Two-Point Boundary Problems, Proceedings of the International Confrence on Statistics and Mathematics and Its Applications in the Development of Science and Technology, (2004), 169-1973.

[21] Othman, M and Abdullah, A.R , The Half-Sweep Multigrid Method as a fast Multigrid Poisson Solver, *Int J.Comp Math* (1998), 319-329

[22] Sulaiman, J, Hasan, M.K and Othman, M, The Half-Sweep Iterative Alternating Decomposition Explicit (HSIADE) Method for Diffusion Equations, *In Comp.l and Info Science* (2004).

[23] Yousif, W and Evans, D.J, Explicit De-coupled Group Iterative Methods and Their Implementations, Parallel Algorithm and Applications (1995), 53-71

[24] Abdulah, A.R, The Four Point Explicit Decoupled group (EDG) Method: A Fast Poisson Solver, *Int.J Comp.Math* (1991),61-70.