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Abstract

The purpose of this study is to apply half-sweep iteration concept with SOR iterative method namely half-sweep SOR (HSSOR) method in solving space-fractional diffusion equations. The Caputo's derivative operators and implicit discretization scheme based on finite difference (FD) approach will be used to approximate linear space-fractional diffusion equations for constructing system of linear equations. Two numerical tests were carried out to show the effectiveness of the proposed method. Then, the results indicated that the HSSOR iterative method has less number of iterations (K) and computational time (time) as compared with FSSOR method. However in term of the maximum error analysis, HSSOR method is comparable with FSSOR

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method. Finally, it can be pointed out that the HSSOR is superior to FSSOR iterative method.

1. Introduction

Recently, fractional derivatives have been considered in biology [1], chemistry [2] and finance [3]. In fact, numerical methods to solve fractional derivative equation problem have been proposed. Therefore, in this paper, Caputo's derivative operator and the implicit FD discretization scheme will be studied to form a reliable algorithm for solving space-fractional diffusion equations. The space-fractional diffusion equations are types of fractional partial differential equations solutions for which are provided by many workers [4, 5]. To do this, the corresponding linear system can be made by using the Caputo's implicit FD approximation equation.

Since the crucial characteristic of the large linear system is sparse, iterative methods used natural option to get efficient solutions. Therefore, this paper considers the development of the half-sweep iteration concept which has been inspired by Abdullah [6]. Since the core feature of half-sweep iterations is to decrease the computational density of the generated linear system, several investigations have been extensively carried out in [7-9] to show the ability of the concept of half-sweep iterative methods. In this study, we applied HSSOR iteration concept together with the Caputo's derivative operator and implicit FD scheme discretization to get numerical solution of the proposed equations.

To show the efficiency of the HSSOR method, let us consider space-fractional diffusion equation (SFDE's) which is stated as

$$\frac{\partial V(x, t)}{\partial t} = a(x)\frac{\partial^{\beta} V(x, t)}{\partial x^{\beta}} + b(x)\frac{\partial V(x, t)}{\partial x} + c(x)V(x, t) + f(x, t),$$

$$\beta > 0, \quad x > 0 \qquad (1)$$

with initial condition $V(x, 0) = f(x), 0 \leq x \leq \ell,$ and boundary conditions

$$V(0, t) = g_0(t), \quad V(\ell, t) = g_1(t), \quad 0 < t \leq T.$$

Before constructing to the approximation equation of problem (1), there are two common fractional derivative operators such as Riemann-Liouville and Caputo. However, this study deals with the use of the Caputo's fractional partial derivative operator, D^{β} of order-β, which is defined as

$$D^{\beta} f(x) = \frac{1}{\Gamma(m-\beta)} \int_0^x \frac{f^{(m)}(t)}{(x-t)^{\beta-m+1}} dt, \quad \beta > 0 \tag{2}$$

with $m - 1 < \beta \leq m$, $m \in N$, $x > 0$.

2. Half-sweep Caputo's FD Approximation Equations

To discretize problem (1), Caputo's fractional derivative operator in equation (2) and the second order difference scheme will be used to construct the following discrete approximation equation:

$$\frac{\partial^{\beta} V(x_i, t_n)}{\partial x^{\beta}} = \sigma_{\beta, 2h} \sum_{j=0,2,4}^{i-2} (V_{i-j+2,n} - 2V_{i-j,n} + V_{i-j-2,n}) g_j^{\beta}, \tag{3}$$

where $\sigma_{\beta, 2h} = \dfrac{(2h)^{-\beta}}{\Gamma(3-\beta)}$ and $g_j^{\beta} = \left(\dfrac{j}{2} + 1\right)^{2-\beta} - \dfrac{j^{2-\beta}}{2}$, while $\Gamma(\cdot)$ is the gamma function.

By using half-sweep Caputo's implicit finite difference scheme, it can be shown the implicit finite difference approximation equation can be stated

$$b_i^* V_{i-2,n} + (\lambda - c_i^*) V_{i,n} - b_i^* V_{i+2,n}$$

$$- a_i^* \sum_{j=0,2,4}^{i-2} g_j^{\beta} (V_{i-j+2,n} - 2V_{i-j,n} + V_{i-j-2,n}) = f_i, \tag{4}$$

where $a_i^* = a_i \sigma_{\beta, 2h}$, $b_i^* = \dfrac{b_i}{4h}$, $c_i^* = c_i$, $F_i^* = f_{i,n}$, and $f_i = \lambda(V_{i,n-2})$ $+ F_i^*$, for $i = 2, 4, 6, ..., \gamma - 1$; $\gamma = M - 1$, $M = 2^p$, $p \geq 2$.

According to equation (4), this equation is known as the half-sweep

Caputo's implicit finite difference approximation equation which is consistent and second order accuracy in space-fractional. Let us define equation (5) for $n > 3$, we obtain

$$-R_i + \alpha_i V_{i-6,n} + s_i V_{i-4} + p_i V_{i-2,n} + q_i V_{i,n} + r_i V_{i+2,n} = f_i,$$

$$i = 2, 4, 6, \ldots, \gamma - 1, \tag{5}$$

where

$$R_i = a_i^* \sum_{j=6}^{i-2} g_j^{\beta} (V_{i-j+2} - 2V_{i-j,n} + V_{i-j-2,n}), \quad \alpha_i = (-a_i^* g_2^{\beta}),$$

$$s_i = (-a_i^* g_1^{\beta} + 2a_i^* g_2^{\beta}), \quad p_i = (b_i^* - a_i^* g_2^{\beta} + 2a_i^* g_1^{\beta} - a_i^*),$$

$$q_i = (-a_i^* g_1^{\beta} + 2a_i^* + (\lambda - c_i^*)), \quad r_i = (-a_i^* - b_i^*).$$

Referring to interior points in solution domain of problem (1), the approximation equation (5) is considered to form the corresponding linear system in matrix form as

$$A\underset{\sim}{V} = \underset{\sim}{f}. \tag{6}$$

3. Concepts of HSSOR Iterations

As explained in the previous section, the tridiagonal linear system in equation (6), it is clear that the characteristics of its coefficient matrix are of large scale and sparse. Then it will be solved by using HSSOR iterative method. HSSOR is applied to half-sweep iteration concept with SOR iterative method. As we know, the main objective of the half-sweep iteration is to reduce the computational complexities during iteration process. To derive the formulation of HSSOR iteration, suppose that the matrix, A in equation (6) can be expressed as:

$$A = D - M - N, \tag{7}$$

where M, D and N represent strictly lower triangular, diagonal and strictly upper triangular matrices, respectively [10]. Since this iterative method is used to find a solution of the linear system, the formulation of HSSOR iterative method is given in [10-12]:

$$\underset{\sim}{V}^{(k+1)} = (D - \omega M)^{-1}[\omega N + (1-\omega)D]\underset{\sim}{V}^{(k)} + (D - \omega M)^{-1}\omega f, \qquad (8)$$

where $\underset{\sim}{V}^{(k)}$ is unknown vector at kth iteration. During the implementation of point iteration, HSSOR iterative algorithms will be imposed onto the even node points $(i = 2, 4, 6, ..., \gamma - 1)$ until the iterative convergence test is reached. Then other approximate solutions at odd node points will be computed directly [6, 9].

4. Numerical Evaluation

To investigate the performance of the HSSOR iteration, two examples are considered. For the comparison, there are three measurement parameters namely number of iterations (K), computational time in second (time) and the maximum of absolute error (max error) at several values of β such as $\beta = 1.2$, $\beta = 1.5$ and $\beta = 1.8$. Then full-sweep SOR (FSSOR) method acts as the control method. Based on the numerical experiments simulation, the convergence test considered the tolerance error $\varepsilon = 10^{-10}$ at five different length sizes as 128, 256, 512, 1024 and 2048.

Example 1 [4]**.**

$$\frac{\partial V(x, t)}{\partial t} = \Gamma(\beta)x^{0.5}\frac{\partial^{\beta}V(x, t)}{\partial x^{\beta}} + (x^2 + 1)\cos(t+1) - 2x\sin(t+1). \qquad (9)$$

The exact solution for equation (9) is given by

$$V(x, t) = (x^2 + 1)\sin(t+1).$$

Example 2 [10]**.**

$$\frac{\partial V(x, t)}{\partial t} = \Gamma(1.2)x^{\beta}\frac{\partial^{\beta}V(x, t)}{\partial x^{\beta}} + 3x^2(2x-1)e^{-t}. \qquad (10)$$

The exact solution for equation (10) is given by $V(x, t) = x^2(1 - x)e^{-t}$.

Numerical results of two examples performed via the FSSOR and HSSOR iterations are recorded in Tables 1 and 2, respectively.

Table 1. Comparison of iterative methods using Example 1 at $\beta = 1.2, 1.5, 1.8$

M	**Method**	**β = 1.2**			**β = 1.5**			**β = 1.8**		
		K	**Time**	**Max Error**	***K***	**Time**	**Max Error**	***K***	**Time**	**Max Error**
128	**FSSOR**	66	1.36	3.37e-02	205	4.08	6.21e-04	733	14.47	2.42e-02
	HSSOR	46	0.47	2.22e-02	110	0.78	6.99e-04	158	6.01	2.42e-02
256	**FSSOR**	129	10.13	2.44e-02	545	42.29	6.69e-04	1361	107.33	2.39e-02
	HSSOR	77	2.93	2.37e-02	212	17.62	6.77e-04	479	49.23	2.39e-02
512	**FSSOR**	278	85.9	2.47e-02	1459	144.73	5.35e-04	3472	725.25	2.37e-02
	HSSOR	129	19.47	2.44e-02	550	81.68	6.69e-04	1465	219.71	2.37e-02
1024	**FSSOR**	607	140.00	2.49e-02	3906	756.12	5.13e-04	5539	1259.97	2.36e-02
	HSSOR	278	68.30	2.47e-02	1385	330.55	5.36e-04	2472	770.92	2.36e-02
2048	**FSSOR**	1230	577.00	2.52e-02	6320	3348.68	5.09e-04	13643	3979.18	2.30e-02
	HSSOR	632	202.34	2.49e-02	3345	1096.00	5.13e-04	7625	1321.10	2.30e-02

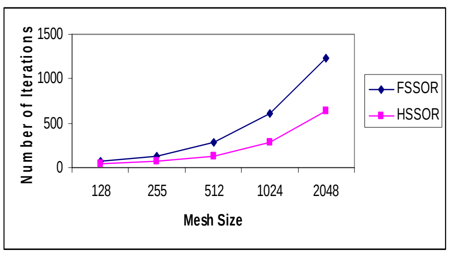
Table 2. Comparison of iterative methods using Example 2 at $\beta = 1.2, 1.5, 1.8$

M	**Method**	**β = 1.2**			**β = 1.5**			**β = 1.8**		
		K	**Time**	**Max Error**	***K***	**Time**	**Max Error**	***K***	**Time**	**Max Error**
128	**FSSOR**	49	1.19	1.80e-01	156	3.77	5.44e-02	332	3.24	1.25e-04
	HSSOR	23	0.43	1.80e-01	56	0.66	5.44e-02	103	1.22	1.22e-04
256	**FSSOR**	103	5.45	1.84e-01	225	14.80	5.58e-02	890	36.00	1.44e-04
	HSSOR	52	2.44	1.84e-01	147	6.69	5.58e-02	313	14.25	1.44e-04
512	**FSSOR**	221	25.31	1.86e-01	732	153.67	5.65e-02	1490	427.00	1.47e-04
	HSSOR	99	13.05	1.86e-01	393	72.47	5.65e-02	661	176.00	1.47e-04
1024	**FSSOR**	271	172.33	5.45e-01	1463	218.00	1.32e-02	4619	2210.72	1.25e-04
	HSSOR	212	52.00	1.89e-01	547	120.00	5.69e-02	3823	1423.03	1.49e-04
2048	**FSSOR**	880	424.00	1.92e-01	2530	953.23	5.73e-02	7710	4120.81	2.30e-04
	HSSOR	495	198.22	1.92e-01	1782	523.00	5.73e-02	5482	2740.23	2.30e-04

5. Conclusion

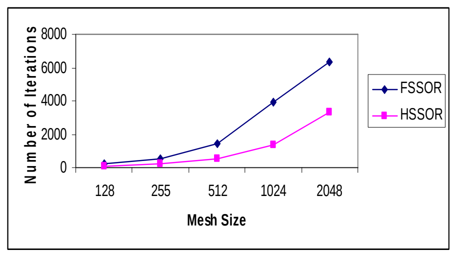
This study presents the half-sweep finite difference approximation equation based on the Caputo's derivative approach and the HSSOR method to solve space-fractional diffusion equations. By referring into Tables 1, 2 and Figures 1, 2, it evidently shows that applications of HSSOR iterative concept can reduce the number of iterations and computational time significantly as compared to FSSOR iterative method. From the observation of results, it can be concluded that both proposed point iterative methods which have been used can be classified under a family of iterative methods. Therefore further investigation of block and two-step iterative methods should be considered for future works.

$\beta = 1.2$



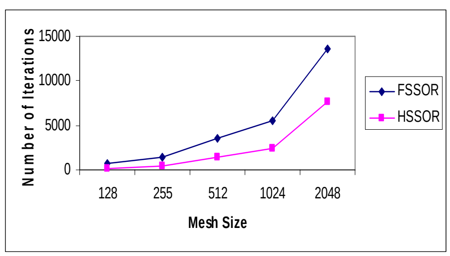
(a)

$\beta = 1.5$



(b)

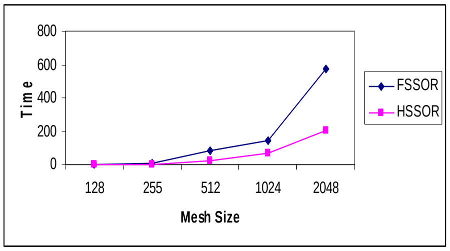
$\beta = 1.8$



(c)

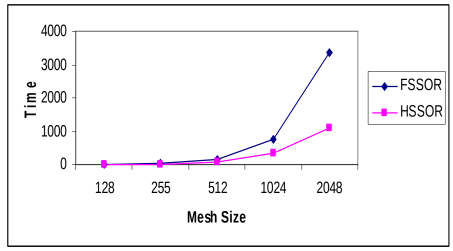
Figure 1. (a), (b) and (c) comparison between number of iterations (K) for FSSOR and HSSOR iterative methods using Example 1 at $\beta = 1.2, 1.5, 1.8$.

$\beta = 1.2$



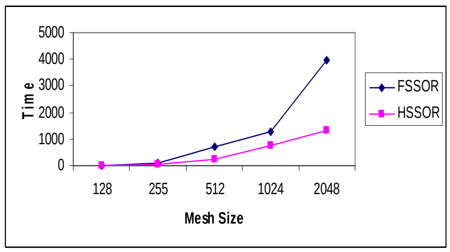
(a)

$\beta = 1.5$



(b)

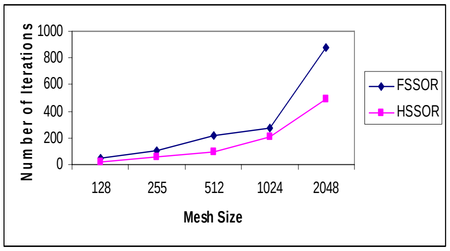
$\beta = 1.8$



(c)

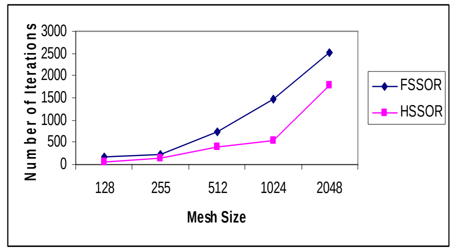
Figure 2. (a), (b) and (c) comparison between time (seconds) for FSSOR and HSSOR iterative methods using Example 1 at $\beta = 1.2, 1.5, 1.8$.

$\beta = 1.2$



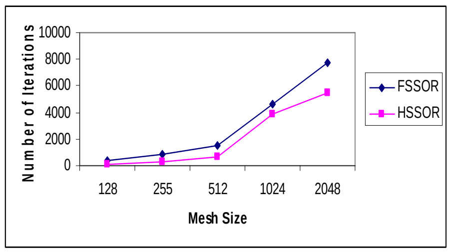
(a)

$\beta = 1.5$



(b)

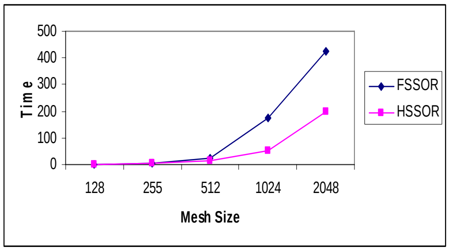
$\beta = 1.8$



(c)

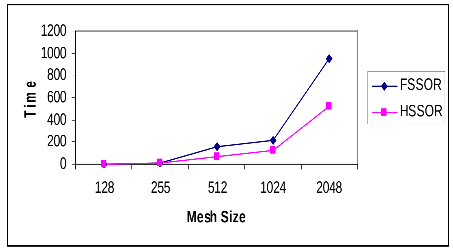
Figure 3. (a), (b) and (c) comparison between number of iterations (K) for FSSOR and HSSOR iterative methods using Example 2 at $\beta = 1.2, 1.5, 1.8$.

$\beta = 1.2$



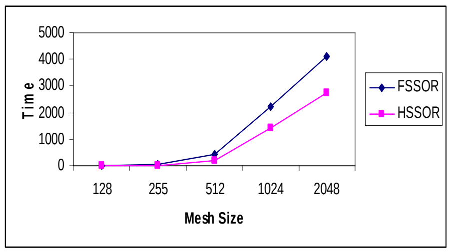
(a)

$\beta = 1.5$



(b)

$\beta = 1.8$



(c)

Figure 4. (a), (b) and (c) comparison between time (seconds) for FSSOR and HSSOR iterative methods using Example 2 at $\beta = 1.2, 1.5, 1.8$.

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