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Performance Analysis of Half-Sweep AOR Method with Nonlocal Discretization Scheme for Nonlinear Two-Point Boundary Value Problem

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Abstract. In this paper, we present the concept of Half-sweep Accelerated OverRelaxation (HSAOR) iterative method with a nonlocal discretization scheme for solving nonlinear two-point boundary value problems. Second order finite difference scheme has been used to derive the half-sweep finite difference (HSFD) approximations of the problems. Then, the nonlocal discretization scheme is applied in order to transform the system of nonlinear approximation equations into the corresponding system of linear equations. Numerical results showed that HSAOR method is superior compared to Full-sweep Gauss-seidel (FSGS), Full-sweep Successive OverRelaxation (FSSOR) and Full-sweep Accelerated Over Relaxation (FSAOR) methods.

INTRODUCTION

Currently, nonlinear two-point boundary value problems have a lot of attention by the researchers. This is because many physical problems in science and engineering can be described mathematically by using the nonlinear equations model. However, it is usually difficult to obtain closed-form solutions either analytically or numerically. In most cases, only approximate solutions can be expected. The nonlinear two-point boundary value problem constantly arise from scientific research, modeling of nonlinear phenomena, chemical reactions and the solution of optimal control problems [1-6]. Therefore, many researchers have been discussed and developed some numerical methods for obtaining approximate solutions to nonlinear two-point boundary value problems such as finite difference method [1], finite element method [2], shooting method [3], spline approximation method [4] and Sinc-Galerkin method [5]. In fact, many studies on various iterative methods have carried out to speed up the convergence rate in solving any system of linear equations. For instance, Young [7, 8, 9], Hackbusch [10] and Saad [11] have already elaborated and discussed the concept of various iterative methods.

Subsequently, Abdullah [12] first introducesthe half-sweep iteration (HSI) concept in solving two-dimensional Poisson equations. This iteration concept is absolutely one of the efficient methods in solving any system of linear algebraic equations. As a matter of fact, the concept of HSI is actually to reduce the computational complexities during iteration process, since the implementation of the method will only consider nearly half of whole node points in a solution domain respectively. In conjunction with this concept, further analysis has beenconducted in [13,14,15,16, 17, 18].

In this paper, however, we analyze the performance of HS iterationconcepts with AOR iterative method namely HSAOR by using the approximation equation based on second order finite difference scheme for solving the proposed problems. Therefore, by imposing the second-order finite difference method, the second-order half-sweep nonlinear

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finite difference approximation equation will be derived to represent the proposed nonlinear problems. Then the approximation equation needs to be linearized over the nonlocal discretization method in order to form a system of linear equations. By the reason of linear system can be generated from the approximation equation, then the numerical solution of the proposed nonlinear problems will be obtained by implementing four proposed point iterative methods such as HSAOR, FSAOR, FSSOR and FSGS iterative methods together with the nonlocal discretization approximation scheme.

To analyze the performance of these four proposed iterative methods, let us consider a nonlinear two-point boundary value problem being defined as

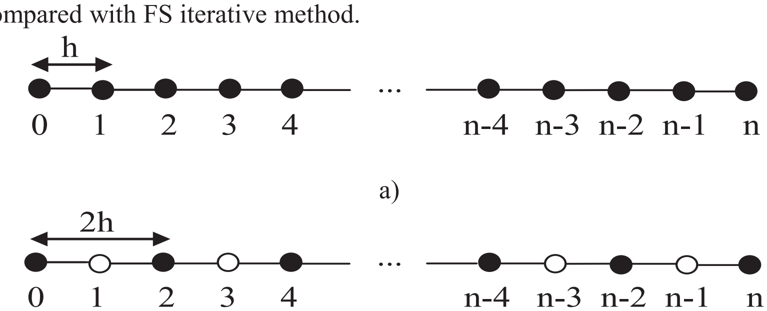
$$-\frac{d^2U}{dx^2} = g(x,U,U') \,,\ a \le x \le b \tag{1}$$

subject to the boundary conditions

$$U(a) = \beta_0, \quad U(b) = \beta_1$$

and β_0, β_1, and $g(x,U)$ are constants and a nonlinear continuous function**,** respectively.

In formulating various iterative schemes such as FS and HS iterations, we need to build the finite grid network as a guide for development and implementation of the proposed methods. Therefore, to derive the formulation of second-order FSFD and HSFD approximation equations, let us consider the finite grid network being used as shown in Figure 1. Then the finite grid network can be used to facilitate us to implement the corresponding proposed algorithms of three iterative methods. According to the point iterations, the implementation of these four iterative methods will be applied onto the node points of the same type until the iterative convergence fixed is achieved. Based on Figure 1, apparently the implementation of the HSiterative method just involves by nearly half of the whole node points as shown in Figure 1(b) compared with FS iterative method.



b)

Figure 1 a) and b) show distribution of uniform solid node points for the FS and HS cases respectively.

Based on Figure 1, the following discussion will be restricted to divide the solution domain of problems into $m = 2^p$, $p \ge 2$ subinternals in which the distance of the subinterval, Δx is defined in Eq. (2).

$$\Delta x = \frac{(b_0 - a_0)}{m} = h,\ n = m - 1 \tag{2}$$

FORMULATION OF NONLOCAL DISCRETIZATION SCHEME

Before constructing the (HSFD) approximation equation of problem (1), let us consider several nonlocal half-sweep discretization schemes being given as follows [19]

$$U_i^2 = U_i U_{i+2} \tag{3}$$

$$U_i^2 = \left(\frac{U_{i-2} + U_{i+2}}{2}\right) U_i \tag{4}$$

$$U_i^3 = \left(\frac{U_{i-2} + U_{i+2}}{2}\right) U_i^2 \tag{5}$$

By using the approach of second-order half-sweep finite difference discretization scheme, the corresponding approximation equations for problem (1) can be easily shown as

$$-U_{i-2} + 2U_i - U_{i+2} - (h)^2 f_i(U_1, U_2, \cdots, U_n) = 0, \quad i = 2, 4, 6 \cdots, n-1 \tag{6}$$

where

$$f_i(U_1, U_2, \cdots, U_n) = g\left(x_i, U_{i,} \frac{U_{i+2} - U_{i-2}}{4(h)}\right) \tag{7}$$

Actually, Eq. (7) is called as the nonlinear term of the problem (1). To solve the nonlinear system in Eq. (6), the nonlocal discretization scheme is used to transform the nonlinear system into the form of a system of linearequations. In this paper, however, we consider the nonlocal discretization scheme in Eq. (4) being imposed over the nonlinear approximation equation (8). Therefore, Eq. (7) can be rewritten as follows

$$f_i(U_1, U_2, \cdots, U_n) = g\left(x_i, \frac{U_{i-2} + U_{i+2}}{2}, \frac{U_{i+2} - U_{i-2}}{4(h)}\right) \tag{8}$$

FORMULATION OF ACCELERATED OVER RELAXATION METHOD

In this section, we present on how to derive the formulation of the HSAOR methods. Therefore, based on the approximation equation in Eqs. (6) and (8), the general scheme of the HSSOR method can be stated as

$$U_i^{(k+1)} = (1-\omega)U_i^{(k)} + \frac{\omega}{2}\begin{pmatrix} U_{i-2}^{(k+1)} + U_{i+2}^{(k)} \\ + h^2 f_i\left(U_1^{(k+1)}, U_2^{(k+1)}, \cdots, U_{i-2}^{(k+1)}, U_{i+2}^{(k)}, \cdots, U_n^{(k)}\right) \end{pmatrix}, i = 2,4,6\cdots, n-1 \tag{9}$$

whereω and $U_i^{(k)}, 2, 4, 6, \cdots n-1$ represent as a relaxation factor and the k[th] represent as a relaxation factor and the k[th] estimation for corresponding exact solutions respectively. Actually the FSSOR iterative method in Eq. (9) was proposed by Young and [20] and [21, 22]. In addition to that, a good choice for the value of the parameter ω can be used to accelerate the convergence rate of the iteration process. In practice, the optimal value of ω in range $1 \le \omega < 2$ will be obtained by implementing several computer programs and to chose the optimum value of ω is chosen in which its number of iterations is the smallest.

Apart from this method, the AOR method which is one of the family of FSSOR presents a two weighted parameters, r and ω as suggested by Hadjidimos [23]. This method can be indicated as FSAOR. These two arbitrary parameters can be fully exploited to produce iterative methods that have faster rates of convergence. Thus, in this paper, we implement the HSAOR iterative method in solving the nonlinear two-point boundary value problems in Eq. (1). Based on Eq. (9), the general scheme for the HSAOR method can be given as

$$\begin{aligned} U_i^{(k+1)} &= (1-\omega)U_i^{(k)} \\ &\quad + \frac{\omega}{2}\begin{pmatrix} U_{i-2}^{(k+1)} + U_{i+2}^{(k)} \\ + h^2 f_i\left(U_1^{(k+1)}, U_2^{(k+1)}, \cdots, U_{i-2}^{(k+1)}, U_{i+2}^{(k)}, \cdots, U_n^{(k)}\right) \end{pmatrix} \\ &\quad + \frac{r}{2}\left(U_{i-2}^{(k+1)} - U_{i-2}^{(k+1)}\right) \end{aligned} \tag{10}$$

for $i = 2, 4, 6 \cdots, n-1$.

According to the Full-sweep case and taking $\omega = r$, the FSAOR method reduces to the FSSOR method, whereas choosing $\omega = r = 1$, this method is called as FSGS method. In this study, the FSGS iterative methods will be used as control methods. The general algorithm for the HSAOR iterative methods to solve the linear equation (9) would be generally described in Algorithm 1.

Algorithm 1 : HSAOR scheme [24]

i. Initialize $U_i^{(0)} \leftarrow 0, \varepsilon \leftarrow 10^{-10}$

ii. Assign the value of ω and r

iii. Calculate $U_i^{(k+1)}$ using

$$U_i^{(k+1)} = (1-\omega)U_i^{(k)} + \frac{\omega}{2}\begin{pmatrix} U_{i-2}^{(k+1)} + U_{i+2}^{(k)} \\ + h^2 f_i\left(U_1^{(k+1)}, U_2^{(k+1)}, \cdots, U_{i-2}^{(k+1)}, U_{i+2}^{(k)}, \cdots, U_n^{(k)}\right) \end{pmatrix} + \frac{r}{2}\left(U_{i-2}^{(k+1)} - U_{i-2}^{(k+1)}\right)$$

iv. Check the convergence test, $\left|U_i^{(k+1)} - U_i^{(k)}\right| \le \varepsilon = 10^{-10}$. If yes, go to step (v). Otherwise go back to step (iii).

v. Display approximate solutions.

NUMERICAL EXPERIMENTS

In order to validate the performance of the HSAOR, FSAOR, FSSOR and FSGS iterative methods together with the nonlocal approach, three nonlinear example problems were tested. For the sake of comparison, three criteria will be considered for these three proposed iterative methods which are number of iterations, execution time (in seconds) and maximum absolute error.

Example 1 [25]

$$\varepsilon y'' + 2y' - e^y = 0, \qquad \text{for} \qquad 0 \le x \le 1 \tag{11}$$

subject to the boundary conditions

$$y(0) = 0 \qquad y(1) = 0$$

with exact solution were defined by

$$y(x) = \log\left(\frac{2}{1+x}\right) - \exp\left(\frac{-2x}{\varepsilon}\right)\log 2. \tag{12}$$

Example 2 [26]

$$y''(x) = \frac{3}{2}y^2, \quad \text{for} \quad 0 < x < 1 \tag{13}$$

subject to the boundary conditions

$$y(0) = -4, \quad y(1) = 1$$

with exact solution were defined by

$$y(x) = \frac{4}{(1+x)^2}. \tag{14}$$

Example 3 [27]

$$y''(x) + \frac{0.5}{x}y'(x) = e^{y(x)}\left(0.5 - e^{y(x)}\right), \quad \text{for} \quad 0 < x < 1 \tag{15}$$

subject to the boundary conditions

$$y(0) = \log[2], \quad y(1) = 0$$

From above three examples, results of numerical experiments obtained have been summarized in Table 1. In the implementation approach, the convergence test considered the tolerance error $\varepsilon = 10^{-10}$.

CONCLUSION

In this paper, the performance of HSAOR method for the solution of nonlinear two-point boundary value problem associated with the second order finite difference approximation scheme has been investigated. Based on Table I, numerical results showed that HSAOR method solved the proposed problems with least number of iterations as compared to the FSGS, FSSOR and FSAOR methods. Meanwhile, in terms of execution time, HSAOR method computes with the fastest time for all considered mesh sizes. In the aspect of accuracy, numerical solutions obtained for test problems 1 to 3 are comparable for all the tested iterative methods. Finally, it can be concluded that the HSAOR method is superior to FSGS, FSSOR and FSAOR methods. This is mainly because of the reduction of computational complexity in which the HSAOR method will only consider approximately half of all interior node points in a solution domain during iteration process.

TABLE 1. Comparison between number of iterations (K), the execution time (seconds) and maximum errors for the iterative methods using example at grid sizes 1024, 2048, 4096, 8192, 16384

	EXAMPLE 1				EXAMPLE 2				EXAMPLE 3			
Number of Iterations												
M	**FSGS**	**FSSOR**	**FSAOR**	**HSAOR**	**FSGS**	**FSSOR**	**FSAOR**	**HSAOR**	**FSGS**	**FSSOR**	**FSAOR**	**HSAOR**
1024	864486	3842	3362	1812	105979	3011	2941	1493	3056978	5751	5127	2565
2048	2867645	7526	6775	3572	3727592	6029	5702	2911	10455079	11819	10248	5126
4096	9106990	14625	13108	7068	452254721	11463	10984	5383	142698383	22356	20487	10246
8192	26969116	27951	26126	13206	796466312	22096	21357	10327	2564789323	49154	40966	20487
16384	70069350	53169	49714	26883	1132546681	42248	40703	20027	3265869745	98306	81923	40966
Execution Time (Seconds)												
M	**FSGS**	**FSSOR**	**FSAOR**	**HSAOR**	**FSGS**	**FSSOR**	**FSAOR**	**HSAOR**	**FSGS**	**FSSOR**	**FSAOR**	**HSAOR**
1024	166.32	1.07	0.85	0.38	173.33	0.57	0.53	0.33	824.02	1.78	1.64	0.71
2048	852.17	2.47	2.13	0.81	816.98	1.41	1.38	0.66	5504.28	6.47	5.82	2.20
4096	4703.43	7.50	6.51	2.25	2374.25	5.32	4.43	1.79	14562.87	24.35	21.93	7.55
8192	25220.13	24.86	23.34	7.08	54236.40	17.85	16.06	5.69	348658.21	95.68	85.11	28.63
16384	40979.70	91.06	84.98	26.36	652315.71	65.21	58.18	20.35	8757731.34	387.77	336.22	110.97
Maximum Errors												
M	**FSGS**	**FSSOR**	**FSAOR**	**HSAOR**	**FSGS**	**FSSOR**	**FSAOR**	**HSAOR**	**FSGS**	**FSSOR**	**FSAOR**	**HSAOR**
1024	1.7868e-02	1.7879e-02	1.7916e-02	1.7955e-02	5.5506e-06	3.6692e-06	3.6907e-06	1.4701e-05	3.3946e-05	8.3448e-07	8.1896e-07	3.3137e-06
2048	1.7837e-02	1.7879e-02	1.7892e-02	1.7907e-02	3.5983e-05	9.3388e-07	8.9782e-07	3.6915e-06	1.3364e-04	2.5406e-07	2.2102e-07	8.7525e-07
4096	1.7711e-02	1.7879e-02	1.7887e-02	1.7887e-02	3.4221e-05	2.7012e-07	1.5937e-07	9.4486e-07	1.3254e-04	1.7409e-07	1.8087e-07	2.9471e-07
8192	1.7209e-02	1.7879e-02	1.7881e-02	1.7885e-02	3.1254e-05	1.4204e-07	1.7234e-07	2.7870e-07	1.3205e-04	1.2937e-08	3.6768e-07	2.1205e-07
16384	1.5208e-02	1.7879e-02	1.7881e-02	1.7880e-02	3.1002e-05	2.1928e-07	3.4580e-07	2.1266e-06	1.3067e-04	3.2070e-09	7.5241e-07	3.7065e-07

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