Application of The Full-Sweep AOR Iteration Concept for Space-Fractional Diffusion Equation

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Abstract. The aim of this paper is to investigate the effectiveness of the Full-Sweep AOR Iterative method by using Full-Sweep Caputo's approximation equation to solve space-fractional diffusion equations. The governing space-fractional diffusion equations were discretized by using Full-Sweep Caputo's implicit finite difference scheme to generate a system of linear equations. Then, the Full-Sweep AOR iterative method is applied to solve the generated linear system. To examine the application of FSAOR method two numerical tests are conducted to show that the FSAOR method is superior to the FSSOR and FSGS methods.

1. Introduction

Many problems in science, industry and engineering can be formulated as mathematical model in a form of fractional partial differential equations (FPDE's) [1,2]. Some different numerical methods have been proposed for solving fractional partial differential equations (FPDE's): Neamaty [3] solved fractional partial differential equations by using wavelet operational method, method of line tranform of the space-fractional Fokker-Plank equations [4] and modified decomposition method for the analytical solution of space fractional diffusion equation [5].

Space-fractional diffusion equations are a type of fractional partial differential equations. Therefore many reaserchers solved the problems numerically. For instance Saadatmandi and Dehghan [6] used a tau approach for solution space-fractional diffusion equation, Feng et al [7] developed second-order approximation for space-fractional diffusion equation with variable coefficient and Sousa [8] applied Spline for solving space-fractional diffusion.

To solve the proposed problem, let us describe some basic definitions and mathematical preliminaries of the fractional derivative theories which are required for our subsequent development.

Definition 1.[9] The Riemann-Liouville fractional integral operator , J^{β} of order-β is defined as

$$J^{\beta}f(x)=\frac{1}{\Gamma(\beta)}\int_{0}^{x}(x-t)^{\beta-1}f(t)dt,\ \beta>0,x>0 \tag{1}$$

Definition 2.[10] The Caputo's fractional partial derivative operator, D^{β} of order -β is defined as

$$D^{\beta}f(x)=\frac{1}{\Gamma(m-\beta)}\int_0^x \frac{f^{(m)}(t)}{(x-t)^{\beta-m+1}}dt,\ \beta>0 \tag{2}$$

with $m-1<\beta\le m, m\in N,\ x>0$. We have following properties when $m-1<\beta\le m, m\in N,\ x>0$:

$D^{\beta}k=0,$ (k is constant)

$$D^{\beta}x^{n}=\begin{cases}0, & \text{for } n\in N_0 \text{ and } n<[\beta]\\ \dfrac{\Gamma(n+1)}{\Gamma(n+1-\beta)}x^{n-\beta}, & \text{for } n\in N_0 \text{ and } n\ge[\beta]\end{cases}$$

where function $[\beta]\le\beta$, $N_0=\{0,1,2,...\}$ and $\Gamma(.)$= Gamma function.

To construct the approximation equation of problem.(1), this work considers implicit finite difference scheme with Full-Sweep Caputo's operator. Both scheme and operator applied over Eq.(3) will derive Full-Sweep Caputo's implicit finite difference approximation equation. Actually this approximation equation has been discussed in constructing a linear system for time-fractional diffusion equations and space-fractional diffusion equations [11,12,13]. Since the linear system will be generated via this approximation equation, this paper consider the Full-Sweep Accelerated Over-Relaxation (AOR) method as linear solver.

According to development of standard AOR iterative method, this methods has been introduced by Hadjidimos [14]. This concept of this method is based on the point iteration concept together with two accelerated parameters. Actually this iterative method is one of the efficient point iterative methods. Due to the advantages of the AOR method for solving linear systems, the aim of this paper is to construct and examine the effectiveness of the FSAOR iterative method for solving space-fractional diffusion equations (SFDE's) based on the Caputo's implicit finite difference approximation equation. To examine the effectiveness of the FSAOR method, we also implement the FSSOR and FSGS iterative methods being used control methods.

Now let us consider the space-fractional diffusion equation being given as :

$$\frac{\partial W(x,t)}{\partial t}=a(x)\frac{\partial^{\beta}W(x,t)}{\partial x^{\beta}}+b(x)\frac{\partial W(x,t)}{\partial x}+c(x)W(x,t)+f(x,t) \tag{3}$$

subject to the initial condition $W(x,0)=f(x),\ 0\le x\le\ell,$ and the Dirichlet boundary conditions $W(0,t)=g_0(t),\quad W(\ell,t)=g_1(t),\ 0<t\le T.$

2. Caputo's Impliciti Finite Difference Approximation for SFDE's

In the section, the process of solving the space-fractional diffusion equations is described in Section 3. Before that let $h=\frac{\ell}{k}$, where k is positive integer. By applying the operator (2), we get for *m=2* that

$$\frac{\partial^{\beta}W(x_i,t_n)}{\partial x^{\beta}}=\sigma_{\beta,h}\sum_{j=0}^{i-1}g_j^{\beta}\left(W_{i-j+1,n}-2W_{i-j,n}+W_{i-j-1,n}\right) \tag{4}$$

where $\sigma_{\beta,h}=\frac{(2h)^{-\beta}}{\Gamma(3-\beta)}$ and $g_j^{\beta}=(j+1)^{2-\beta}-j^{2-\beta}$.

Then by imposing the implicit finite difference scheme and the Caputo's operator in Eq.(4) over Eq.(3), the Full-Sweep Caputo's implicit finite difference scheme can be written as

$$\lambda\left(W_{i,n}-W_{i,n-1}\right)=a_i\sigma_{\beta,h}\sum_{j=0}^{i-1}g_j^{\beta}\left(W_{i-j+1,n}-2W_{i-j,n}+W_{i-j-1,n}\right)+b_i\frac{\left(W_{i+1,n}-W_{i-1,n}\right)}{2h}+C_iW_{i,n}+f_{i,n} \tag{5}$$

for *i = 1,2,4,...m-1*. Then the approximation equation (5) can be simplified again as

$$\lambda W_{i,n-1}=-a_i\sigma_{\beta,h}\sum_{j=0}^{i-1}g_j^{\beta}\left(W_{i-j+1,n}-2W_{i-j,n}+W_{i-j-1,n}\right)-\frac{b_i}{2h}\left(W_{i+1,n}-W_{i-1,n}\right)-C_iW_{i,n}+\lambda W_{i,n}-f_{i,n}$$

(6) After simplifying from Eq.(6), we have

$$b_i^* W_{i-1,n} + (\lambda - c_i^*) W_{i,n} - b_i^* W_{i+1,n_i} - a_i^* \sum_{j=0}^{i-1} g_j^{\beta} \left(W_{i-j+1,n} - 2W_{i-j,n} + W_{i-j-1,n} \right) = f_i \quad (7)$$

where $a_i^* = a_i \sigma_{\beta,2h}$, $b_i^* = \frac{b_i}{2h}$, $c_i^* = c_i$, $F_i^* = f_{i,n}$, $f_i = \lambda \left(W_{i,n-2} \right) + F_i^*$.

By referring to Eq.(7), be expanded to get the following approximation equation

$$R_i + \alpha_i W_{i-3,n} + s_i W_{i-2,n} + p_i W_{i-1,n} + q_i W_{i,n} + r_i W_{i+1,n} = f_i \quad (8)$$

where

$$R_i = a_i^* \sum_{j=3}^{i-1} g_j^{\beta} \left(W_{i-j+1} - 2W_{i-j,n} + W_{i-j-1,n} \right), \quad \alpha_i = \left(-a_i^* g_2^{\beta} \right), \quad s_i = \left(-a_i^* g_1^{\beta} + 2a_i^* g_2^{\beta} \right),$$

$$p_i = \left(b_i^* - a_i^* g_2^{\beta} + 2a_i^* g_1^{\beta} - a_i^* \right), \quad q_i = \left(-a_i^* g_1^{\beta} + 2a_i^* + (\lambda - c_i^*) \right), \quad r_i = \left(-a_i^* - b_i^* \right).$$

Thus by referring Eq.(8), we can develop to a linear system in matrix form as

$$A \underset{\sim}{W} = \underset{\sim}{f} \quad (9)$$

3. Full-Sweep AOR Iterative Method

By considering linear system in Eq.(9), it is clear that the characteristics of its coefficient matrix is large scale and sparse. In previous studies, a lot of works have been done to establish various iterative methods such as Young [15], Hackbusch [16], Saad [17]. To solve the linear system which is generated by the Caputo's implicit finite difference approximation equation, we consider the FSAOR method which is known as the Standard Accelerated Over-Relaxation (AOR) iterative method [11]. This iterative method is the most known and widely using for solving any linear systems [12,13]. To derive the formulation of FSAOR method in matrix form, we consider the coefficient matrix *A* (9) being expressed as summation of the three matrices for FSAOR method

$$A = D - L - U \quad (10)$$

where *D*, *L* and *U* are diagonal, lower triangular and upper triangular matrices respectively. From Eq.(10), the general scheme for FSAOR iterative method can be written as [14,18]

$$\tilde{W}^{(k+1)} = (D - \omega L)^{-1} [\beta U + (\beta - \omega) D + (1 - \beta) D] \tilde{W}^{(k)} + \beta (D - \omega L)^{-1} f \quad (11)$$

where $\tilde{V}^{(k)}$ represents an unknown vector at k^{th} iteration. As $\beta = \omega$, this iterative method (11) will be named as the FSSOR [14,15], while $\beta = \omega = 1$ we get the FSGS method. Based on Eq.(11), the general algorithm for FSAOR iterative method to solve linear system (9) would be generally described in Algorithm 1.[11]

Algorithm 1: FSAOR method

i. Initialize $\tilde{V} \leftarrow 0$ and $\varepsilon \leftarrow 10^{-10}$.

ii. For $j = 1,2,\ldots,n-1$ implement

For $i = 1,2,\ldots,m-1$ calculate $\tilde{W}^{(k+1)} = (D - \omega L)^{-1} [\beta U + (\beta - \omega) D + (1 - \beta) D] \tilde{W}^{(k)} + \beta (D - \omega L)^{-1} f$

iii Convergence test. If the convergence criterion i.e $\left\| \tilde{W}^{(k+1)} - \tilde{W}^{(k)} \right\| \leq \varepsilon = 10^{-10}$ is satisfied, go to Step (iii). Otherwise go back to Step (ii).

iv Display approximate solutions.

4. Numerical Experiment

In this section , two numerical examples are illustrated to show acccuray and effectiveness of the proposed method. Three criteria will be considered in comparison for FSGS, FSSOR and FSAOR such as K (number of iterations), Time (execution time in seconds) and Max Error (maximum error) at $\beta = 1.2,\ \beta = 1.5$ and $\beta = 1.8$ and several mesh sizes as 128, 256, 512,1024 and 2048. During the implementation of numerical experiments, the convergence test considered the tolerance error $\varepsilon = 10^{-10}$. The results of numerical simulations which were obtained from applications of the FSGS, FSSOR and FSAOR iterative methods for the following examples 1 and 2 have been recorded in Tables 1 and 2 respectively.

Example 1 : [10]

$$\frac{\partial w(x,t)}{\partial t} = d(x)\frac{\partial^{\beta} w(x,t)}{\partial x^{\beta}} + p(x,t), \tag{12}$$

The exact solution of problem (12) is $w(x,t) = (x^2+1)\sin(t+1)$.

Examples 2 : [10]

$$\frac{\partial w(x,t)}{\partial t} = \Gamma(1.2)x^{\beta}\frac{\partial^{\beta} w(x,t)}{\partial x^{\beta}} + 3x^2(2x-1)e^{-t}, \tag{13}$$

The exact solution of this problem is $w(x,t) = x^2(1-x)e^{-t}$.

TABLE 1. Comparison between K (number of iterations), TIME (the execution time in seconds) and Max Error (maximum errors) for the iterative methods using Example 1 at $\beta = 1.2, 1.5, 1.8$

M	Method	β = 1.2			β = 1.5			β = 1.8		
		K	Time (Seconds)	Max Error	K	Time (Seconds)	Max Error	K	Time (Seconds)	Max Error
128	**FSGS**	74	1.48	2.37e-02	251	4.95	6.20e-04	930	18.29	3.99e-02
	FSSOR	66	1.36	2.37e-02	205	4.08	6.21e-04	733	14.47	2.42e-02
	FSAOR	65	1.32	2.37e-02	188	3.88	6.21e-04	269	5.35	3.99e-02
256	**FSGS**	152	11.64	2.44e-02	666	51.01	5.69e-04	3029	233.01	3.97e-02
	FSSOR	129	10.13	2.44e-02	545	42.29	6.69e-04	1361	107.33	2.39e-02
	FSAOR	128	10.00	2.44e-02	370	28.88	5.69e-04	756	58.90	3.97e-02
512	**FSGS**	352	99.64	2.47e-02	1780	550.52	5.36e-04	9840	755.31	3.96e-02
	FSSOR	278	85.9	2.47e-02	1459	144.73	5.35e-04	3472	725.25	2.37e-02
	FSAOR	270	84.05	2.47e-02	983	104	5.35e-04	2497	703	3.96e-02
1024	**FSGS**	709	672.27	2.49e-02	4750	1870.68	5.13e-04	21847	5259.97	3.95e-02
	FSSOR	607	140	2.49e-02	3906	756.12	5.13e-04	5539	1259.97	2.36e-02
	FSAOR	577	125	2.49e-02	3640	689	5.13e-04	5220	1119	2.36e-02
2048	**FSGS**	1547	1227.21	2.50e-02	8320	4348.68	5.02e-04	47322	8979.18	3.93e-02
	FSSOR	1230	577.00	2.52e-02	6320	3348.68	5.09e-04	13643	3979.18	2.30e-02
	FSAOR	1150	540	2.52e-02	5950	3102	5.09e-04	13203	3920	2.30e-02

TABLE 2. Comparison between K (number of iterations), TIME (the execution time in seconds) and Max Error (maximum errors) for the iterative methods using Example 2 at $\beta = 1.2, 1.5, 1.8$

M	Method	β = 1.2			β = 1.5			β = 1.8		
		K	Time (Seconds)	Max Error	K	Time (Seconds)	Max Error	K	Time (Seconds)	Max Error
128	**FSGS**	57	1.42	1.80e-01	182	4.41	5.44e-02	569	13.70	8.88e-04
	FSSOR	49	1.19	1.80e-01	156	3.77	5.44e-02	332	3.24	1.25e-04
	FSAOR	48	0.93	1.80e-01	133	1.41	5.44e-02	148	1.52	1.25e-04
256	**FSGS**	117	10.95	1.84e-01	481	45.32	5.58e-02	931	174.77	4.09e-04
	FSSOR	103	5.45	1.84e-01	225	14.80	5.58e-02	890	36.00	1.44e-04
	FSAOR	97	3.58	1.84e-01	197	10.93	5.58e-02	457	16.66	1.44e-04
512	**FSGS**	221	25.31	1.86e-01	732	153.67	5.65e-02	1635	427	1.47e-04
	FSSOR	128	20.69	5.39e-01	553	86.22	1.28e-02	1490	374.84	1.54e-04
	FSAOR	106	18.71	5.39e-01	525	83.02	1.28e-02	1357	193.83	1.53e-04
1024	**FSGS**	480	115.89	1.89e-01	1923	714.51	5.69e-02	5937	948.83	1.49e-04
	FSSOR	271	172.33	5.45e-01	1463	218	1.32e-02	4619	2210.72	1.25e-04
	FSAOR	213	168	5.45e-01	1298	198	1.32e-02	4329	2103	1.25e-04
2048	**FSGS**	1186	557.00	1.88e-01	6241	1259.31	5.85e-02	8482	5345.02	1.20e-04
	FSSOR	880	424.00	1.92e-01	2530	953.23	5.73e-02	7710	4120.81	2.30e-04
	FSAOR	815	398	1.92e-01	2506	912	5.73e-02	6520	3834	2.30e-04

5. Conclusion

This paper investigates the effectiveness the FSAOR iterative method to get numerical solutions of space-fractional diffusion equations. Throught the results obtained for examples 1 and 2, it manifestly shows that the application of FSAOR iteration which uses two accelerated parameters can reduce number of iterations and computational time as shown in Tables 1 and 2. Again this accuracy of numerical solutions of this method is comparable with the FSSOR and FSGS method. Since the FSAOR method is one of Full-Sweep iteration family, our future work will be extended to investigate the effectiveness of the Half-Sweep iteration family [19,20].

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