Australian Journal of Basic and Applied Sciences, 8(24) Special 2014, Pages: 153-158

AENSI Journals
Australian Journal of Basic and Applied Sciences
ISSN:1991-8178

Journal home page: www.ajbasweb.com

Full-Sweep SOR Iterative Method to Solve Space-Fractional Diffusion Equations

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ARTICLE INFO

Article history:
Received 30 September 2014
Received in revised form
17 November 2014
Accepted 25 November 2014
Available online 13 December 2014

Keywords:
Caputo's Space-Fractional, Implicit finite difference, FSSOR method

ABSTRACT

Background: Space-Fractional diffusion equations are generalizations of classical diffusion equations which are used in modeling superdiffusive problem. In this paper, we examine the effectiveness of the FSSOR iterative method to solve space-fractional diffusion equations based on an unconditionally implicit finite difference scheme. From the discretization of the one dimensional linear space-fractional diffusion equations by using the Caputo's space fractional derivative. We can derive the Caputo's implicit finite difference approximation equation. Then this approximation equations hence will be used to generated the corresponding system of linear equations. Two numerical examples were used to make a comparison between FSGS and FSSOR methods.Based on computational numerical results, it can be concluded that the proposed FSSOR iterative method is superior to the FSGS iterative method

To Cite This Article: Andang Sunarto, Jumat Sulaiman, Azali Saudi, Full-Sweep SOR Iterative Method To Solve Space-Fractional Diffusion Equations. ***Aust. J. Basic & Appl. Sci.,*** *8(24): 153-158, 2014*

INTRODUCTION

In recent years, many researchers are concerned to study of fractional partial differential equations[(Choi *et al*.,2010), (Tadjeran *et al*.,2006), (Mohammad *et al*.,2014)]. The space-fractional diffusion equations are type of fractional partial differential equations in which many researcher are solved numerically. For getting solution of these problems, many researchers were seeking different ways to solve these problems. For instance (Azizi *et al*.,2013) used Chebyshev collocation method to discretize space-fractional to obtain a linear system of ordinary differential equations and used the finite difference method for solving the resulting system. Then (Saadatmandi *et al*.,2011) used tau approach, while (Shen *et al*., 2005) used insulated ends and explicit finite difference to solve space-fractional diffusion equations. Also (Aslefallah *et al*.,2014) applied theta scheme to solve space-fractional diffusion equations.

In this paper, we propose a different approach to get numerical solutions of the one-dimensional space-fractional partial differential equation (SFPDE's) iteratively which defined as

$$\frac{\partial U(x,t)}{\partial t} = a(x)\frac{\partial^{\beta} U(x,t)}{\partial x^{\beta}} + b(x)\frac{\partial U(x,t)}{\partial x} + c(x)U(x,t) + f(x,t) \tag{1}$$

with initial condition
$U(x,0) = f(x),\ 0 < x < \ell,$

and boundary conditions
$U(0,t) = g_0(t),\ 0 < t \le T,$
$U(\ell,t) = g_1(t),\ 0 < t \le T.$

Before developing the discrete equation of Problem (1), we describe some necessary basic definitions for fractional derivatives which are used in the paper.

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Definition 1.(Zhang, 2009) The Riemann-Liouville fractional integral operator , J^{β} of order-β is defined as

$$J^{\beta}f(x)=\frac{1}{\Gamma(\beta)}\int_0^x (x-t)^{\beta-1}f(t)dt,\ \ \beta>0,\ x>0 \qquad (2)$$

Definition 2.(Azizi *et al*.,2013) The Caputo's fractional partial derivative operator, D^{β} of order -β is defined as

$$D^{\beta}f(x)=\frac{1}{\Gamma(m-\beta)}\int_0^x \frac{f^{(m)}(t)}{(x-t)^{\beta-m+1}}dt,\ \ \beta>0 \qquad (3)$$

with $m-1<\beta\le m,\ m\in N,\ x>0$

We have the following properties when $m-1<\beta\le m,\ x>0$:

$D^{\beta}k=0,\ (k \text{ is a constant}),$

$$D^{\beta}x^n=\begin{cases}0, & \text{for } n\in N_0 \text{ and } n<[\beta]\\ \dfrac{\Gamma(n+1)}{\Gamma(n+1-\beta)}x^{n-\beta}, & \text{for } n\in N_0 \text{ and } n\ge[\beta]\end{cases}$$

where function $[\beta]$ to denote the smallest integer greater than or equal to β, $N_0=\{0,1,2,...\}$ and $\Gamma(.)$ is the gamma function.

According to previous studies, many studies have been conducted to investigate the effectiveness of the FSSOR method [(Youssef, 2012), (Sun, 2005), (Starke *et al*., 1991), (Hadjidimos, 2000)]. However, there is no study that have been conducted in literature for solving space-fractional diffusion equation Problem (1) by FSSOR method. This paper make an effort to examine the effectiveness Full-Sweep Successive Over-Relaxation (FSSOR) iterative method which is compared with the Full-Sweep Gauss-Seidel (FSGS) iterative method to solve Problem (1). For the numerical solution of Problem (1), we derive numerical approximation equation based on the Caputo's space-fractional derivative definition with Dirichlet boundary conditions and also consider the non-local fractional derivative operator. Derivation of Caputo's implicit finite difference approximation equation can be categoried as unconditionally stable scheme.

Derivation of Caputo's Implicit Finite Difference Approximation:

Assume that $h=\frac{\ell}{k}$, k is positive integer and using second order approximation, we get

$$\frac{\partial^{\beta}U(x_i,t_n)}{\partial x^{\beta}}=\frac{1}{\Gamma(2-\beta)}\int_0^{t_n}\frac{\partial^2 U(x_i,s)}{\partial x^2}(t_n-s)^{1-\beta}\partial s$$

$$=\frac{1}{\Gamma(2-\beta)}\sum_{j=0}^{i-1}\int_{jh}^{(j+1)h}\left(\frac{U_{i-j+1,n}-2U_{i-j,n}+U_{i-j-1,n}}{h^2}\right)(nh-s)^{\beta}\partial s$$

$$=\frac{h^{-\beta}}{\Gamma(3-\beta)}\sum_{j=0}^{i-1}\left(U_{i-j+1,n}-2U_{i-j,n}+U_{i-j-1,n}\right)\left((j+1)^{2-\beta}-j^{2-\beta}\right) \qquad (4)$$

Let us define

$$\sigma_{\beta,h}=\frac{h^{-\beta}}{\Gamma(3-\beta)}$$

and

$$g_j^{\beta}=(j+1)^{2-\beta}-j^{2-\beta}$$

then the discrete approximation of Eq. (4)

$$\frac{\partial^{\beta}U(x_i,t_n)}{\partial x^{\beta}}=\sigma_{\beta,h}\sum_{j=0}^{i-1}g_j^{\beta}\left(U_{i-j+1,n}-2U_{i-j,n}+U_{i-j-1,n}\right)+O(i)$$

Now we approximate Problem (1) by using Caputo's implicit finite difference approximation :

$$\lambda(U_{i,n}-U_{i,n-1})=a_i\sigma_{\beta,h}\sum_{j=0}^{i-1}g_j^{\beta}\left(U_{i-j+1,n}-2U_{i-j,n}+U_{i-j-1,n}\right)+b_i\frac{(U_{i+1,n}-U_{i-1,n})}{2h}$$

$$+C_iU_{i,n}+f_{i,n}$$

for $i=1,2,\ldots,m-1$. Then we can simplify the scheme approximation equation as

$$\lambda U_{i,n-1} = -a_i\sigma_{\beta,h}\sum_{j=0}^{i-1} g_j^{\beta}\left(U_{i-j+1,n} - 2U_{i-j,n} + U_{i-j-1,n}\right) - \frac{b_i}{2h}\left(U_{i+1,n} - U_{i-1,n}\right) - C_i U_{i,n} + \lambda U_{i,n} - f_{i,n}$$

So we get :

$$\therefore b_i^* U_{i-1,n} + \left(\lambda - c_i^*\right)U_{i,n} - b_i^* U_{i+1,n} - a_i^*\sum_{j=0}^{i-1} g_j^{\beta}\left(U_{i-j+1,n} - 2U_{i-j,n} + U_{i-j-1,n}\right) = f_i \quad (5)$$

where

$$a_i^* = a_i\sigma_{\beta,h}\ ,\ b_i^* = \frac{b_i}{2h}\ ,\ c_i^* = c_i\ ,\ F_i^* = f_{i,n}$$

and $f_i = \lambda\left(U_{i,n-1}\right) + F_i^*$

According to Eq. (5), the approximation equation is knows as the fully implicit finite difference approximation equation which is consistent second order accuracy in space-fractional. For simplicity, let Eq. (5) for $n > 3$ be rewritten as

$$-R_i + \alpha_i U_{i-3,n} + s_i U_{i-2} + p_i U_{i-1,n} + q_i U_{i,n} + r_i U_{i+1,n} = f_i \quad (6)$$

where

$$R_i = a_i^*\sum_{j=3}^{i-1} g_j^{\beta}\left(U_{i-j+1} - 2U_{i-j,n} + U_{i-j-1,n}\right),$$
$$\alpha_i = \left(-a_i^* g_2^{\beta}\right),$$
$$s_i = \left(-a_i^* g_1^{\beta} + 2a_i^* g_2^{\beta}\right),$$
$$p_i = \left(b_i^* - a_i^* g_2^{\beta} + 2a_i^* g_1^{\beta} - a_i^*\right),$$
$$q_i = \left(-a_i^* g_1^{\beta} + 2a_i^* + \left(\lambda - c_i^*\right)\right),$$
$$r_i = \left(-a_i^* - b_i^*\right).$$

Then Eq. (6) can be used to construct a linear system in matrix form as

$$A\underset{\sim}{U} = \underset{\sim}{f} \quad (7)$$

where

$$A = \begin{bmatrix} p_1* & q_1* & r_1* & & & & \\ & p_2* & q_2* & r_2* & & & \\ & & p_3* & p_3* & r_3* & & \\ & & & \ddots & \ddots & \ddots & \\ & & & & p_{m-2}* & q_{m-2}* & r_{m-2}* \\ & & & & & p_{m-1} & q_{m-1} \end{bmatrix}_{(m-1)x(m-1)}$$

$$\underset{\sim}{U} = \begin{bmatrix} U_{11} & U_{21} & U_{31} & \cdots & U_{m-2,1} & U_{m-1,1} \end{bmatrix}^T,$$

$$\underset{\sim}{f} = \begin{bmatrix} U_{11} + p_1 U_{01} & U_{21} & U_{31} & \cdots & U_{m-2,1} & U_{m-1,1} + p_{m-1}U_{m,1} \end{bmatrix}^T.$$

Full-Sweep Successive Over-Relaxation Iterative Method:

Based on the tridiagonal linear system in Eq. (7), it is clear that the characteristics of its coefficient matrix has large scale and sparse. Essentially, the various concept of iterative methods has been conducted by many researchers such, [(Young, 1954,1971,1972), (Hackbush, 1995), (Saad, 1996), (Evans, 1985), (Yousif and Evans, 1995) and (Othman and Abdullah, 2000)]. For solving the tridiagonal linear system, (Young, 1954,1971,1972) initiated Full-Sweep Successive Over-Relaxation method. Namely as standard SOR is the most known and widely used iterative techniques to solve in solving any linear systems. To derive this FSSOR method, let the coefficient matrix A in (7) can be expressed as summation of the three matrices

$$A = D - L - V$$

where D, L, and V are diagonal, lower triangular and upper triangular matrices respectively. Then, Sor iterative method can be defined generally as

$$\underset{\sim}{U}^{(k+1)} = (D - \omega L)^{-1}[\omega V + (1-\omega)D]\,\underset{\sim}{U}^{(k)} + (D - \omega L)^{-1} f \qquad (8)$$

where $\underset{\sim}{U}^{(k)}$ represents an unknown vector at k^{th} iteration. The application of the FSSOR method can be described in Algorithm 1.

Algorithm 1: FSSOR Method:

i. Initialize $\underset{\sim}{U} \leftarrow 0$ and $\varepsilon \leftarrow 10^{-10}$

ii. For i=1,2,…,n-1 and j=1,2,…,m-1 assign
$\underset{\sim}{U}^{(k+1)} = (D - \omega L)^{-1}[\omega V + (1-\omega)D]\,\underset{\sim}{U}^{(k)} + (D - \omega L)^{-1} f$

iii. Convergence test. If the convergence criterion i.e
$\left\| \underset{\sim}{U}^{(k+1)} - \underset{\sim}{U}^{(k)} \right\| \le \varepsilon = 10^{-10}$ is satisfied, go step (iv)
Otherwise go back to step (ii)

iv. Display approximate solutions

Numerical Experiment:

In this section, we have selected two examples of the space-fractional diffusion equations to verify the effectiveness of the Full-Sweep Gauss-Seidel (FSGS) and Full-Sweep Successive Over-Relaxation (FSSOR) iterative methods. In comparison, three criteria will be considered for both iterative methods such as number iterations, the execution time (seconds) and maximum error at three different values of $\beta = 1.2$, $\beta = 1.5$ and $\beta = 1.8$. During the implementation of the point iterations, the convergence test considered the tolerance error, $\varepsilon = 10^{-10}$.

***Examples 1** (Azizi et al., 2013):*

Let us consider the following space-fractional initial boundary value problem

$$\frac{\partial U(x,t)}{\partial t} = d(x)\frac{\partial^{1.5} U(x,t)}{\partial x^{1.5}} + p(x,t), \qquad (9)$$

On finite domain $0 < x < 1$, with the diffusion coefficient $d(x) = \Gamma(1.5)x^{0.5}$,
the source function $p(x,t) = (x^2+1)\cos(t+1) - 2x\sin(t+1)$,
with the initial condition $U(x,0) = (x^2+1)\sin(1)$
and the boundary conditions $U(0,t) = \sin(t+1)$, $U(1,t) = 2\sin(t+1)$, for t>0.
The Exact solution of this problem is $U(x,t) = (x^2+1)\sin(t+1)$

Examples 2 (Azizi *et al*., 2013) **:**

Let us consider the following space-fractional initial boundary value problem

$$\frac{\partial U(x,t)}{\partial t} = \Gamma(1.2)x^{1.8}\frac{\partial^{1.8} U(x,t)}{\partial x^{1.8}} + 3x^2(2x-1)e^{-t}, \qquad (10)$$

with the initial condition
$U(x,0) = x^2 - x^3$, and zero Dirichlet conditions
The exact solution of this problem is $U(x,t) = x^2(1-x)e^{-t}$

All numerical results for problems (9) and (10), obtained from application of FSGS and FSSOR iterative methods are recorded in Tables 1 and 2 by using the different value of mesh size, M=128, 256, 512, 1024 and 2048.

Table 1: Comparison of number iterations, the execution time (seconds) and maximum errors for the iterative methods using example at $\beta = 1.2, 1.5, 1.8$

M	Method	β = 1.2			β = 1.5			β = 1.8		
		K	Time	Max Error	K	Time	Max Error	K	Time	Max Error
128	FSGS	62	1.34	2.06e-02	205	4.15	1.02e-02	757	15.11	5.28e-02
	FSSOR	25	0.12	2.06e-02	95	1.56	1.01e-02	311	6.01	5.28e-02
256	FSGS	73	5.66	9.71e-02	299	23.20	3.23e-02	1361	107.33	6.63e-02

	FSSOR	30	2.22	9.71e-02	112	9.25	3.22e-02	432	49.23	6.63e-02
512	FSGS	84	26.18	1.79e-01	430	132.51	4.26e-02	2411	755.31	5.71e-02
	FSSOR	34	9.21	1.79e-01	180	55.38	4.26e-02	941	226.24	5.71e-02
1024	FSGS	96	120.29	2.42e-01	613	755.97	4.82e-02	4221	5259.97	5.23e-02
	FSSOR	42	60.08	2.42e-01	211	330.55	4.82e-02	1095	1201.30	5.23e-02
2048	FSGS	109	577.00	2.85e-01	866	4348.68	5.09e-02	7322	4979.18	5.09e-02
	FSSOR	49	202.34	2.85e-01	396	1096.00	5.08e-02	2625	1121.10	5.09e-02

Table 2: Comparison of number iterations, the execution time (seconds) and maximum errors for the iterative methods using example at $\beta = 1.2, 1.5, 1.8$

M	Method	β = 1.2			β = 1.5			β = 1.8		
		K	Time	Max Error	K	Time	Max Error	K	Time	Max Error
128	FSGS	48	1.19	1.82e-01	150	3.77	6.01e-02	473	11.36	6.31e-03
	FSSOR	19	0.28	1.82e-01	85	1.23	6.01e-02	189	4.95	6.31e-03
256	FSGS	57	5.45	1.77e-01	225	21.61	7.64e-02	890	85.00	2.88e-02
	FSSOR	23	2.01	1.77e-01	98	9.11	7.64e-02	394	39.25	2.88e-02
512	FSGS	67	25.31	1.52e-01	331	124.05	8.70e-02	1635	619.64	5.02e-02
	FSSOR	27	9.15	1.52e-01	122	54.75	8.70e-02	461	212.10	5.02e-02
1024	FSGS	77	115.89	1.20e-01	477	714.51	9.00e-02	2937	4448.83	6.75e-02
	FSSOR	33	52.00	1.20e-01	190	320.00	9.00e-02	983	1123.03	6.75e-02
2048	FSGS	88	557.00	1.04e-01	679	4259.31	9.25e-02	5171	55209.81	7.92e-02
	FSSOR	39	198.22	1.04e-01	230	1232.00	9.25e-02	2126	19807.23	7.92e-02

Conclusion:

In this paper we proposed a Caputo’s implicit finite difference and scheme FSSOR method to solve the space-fractional diffusion equations. We have applied the formulation of the Caputo’s finite difference equations to generate a corresponding linear system. Then for solving the linear system, the formulation of FSGS and FSSOR iterative methods have been constructed based on the Caputo’s derivative operator. From observation of all experimental results by imposing the FSGS and FSSOR iterative methods, it can be also observed in tables 1 and 2 that the number of iterations and the execution time for FSSOR iterative method have been declined tremendously as compared with FSGS iterative method. This is due to the implementations of FSSOR iterative method have been accelerated by using the optimal value of the weighted parameter, ω. According to the accuracy and of both iterative methods, it can be concluded that their numerical solutions are in good agreement.

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