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QSAOR Iterative Method for the Solution of Time-Fractional Diffusion Equation

[1]Baharuddin, [2]Andang Sunarto and [3]Juhriyansyah Dalle
[1]State University of Medan, Medan, Indonesia
[2]State Institute for Islamic Studies of Bengkulu, Bengkulu, Indonesia
[3]University of Lambung Mangkurat, Banjarmasin, Indonesia

Abstract: The main purpose of this study is to investigate the performance of the Quarter-Sweep AOR (QSAOR) iterative method by using Caputo fractional derivative operator and implicit finite difference scheme to solve time-fractional diffusion equation. To solve the problems, a linear system will be constructed via discretization of the one-dimensional linear time-fractional diffusion equations by using the Caputo's fractional derivative. Then the generated linear system has been solved using the proposed QSAOR iterative method. In the addition, the formulation and application of the QSAOR method to solve the problems are also presented. Two numerical examples and comparison with FSAOR and HSAOR methods are given to show the effectiveness of QSAOR method.

Key words: QSAOR, Caputo, implicit, time-fractional, diffusion

INTRODUCTION

Many science and engineering problems can be modeled mathematically as differential equations (Oldham and Spanier, 1974; Podlubny, 1999a). Since, the rapid growth of computer technology, the numerical techniques are used to solve a large size problem. Then, there are some numerical methods proposed to solve the Time Fractional Diffusion Equations (TFDE) such as Crank-Nicolson finite difference method (Sweilam *et al*., 2012), two implicit finite method (Ma, 2014) and high-order finite element methods (Jiang and Ma, 2011). In this research, one dimensional time-fractional diffusion equation will be solved numerically and represented as follows:

$$\frac{\partial^{\alpha}U(x,t)}{\partial x^{\alpha}} = a(x)\frac{\partial^2 U(x,t)}{\partial x^2} + b(x)\frac{\partial U(x,t)}{\partial x} + c(x)U(x,t) \quad (1)$$

With boundary conditions $u(0, t) = g_0(t)$, $U(l, t) = g_1(t)$ and the initial condition $U(x, 0) = f(x)$. To get the approximate solution of the Time Fractional Diffusion Equation (TFDE) problem the problem needs to be discretized to form an approximation equation. Based on the implicit finite difference scheme and Caputo fractional derivative operator, the approximation equations can be derived to construct a linear system at each time level. For to solving linear systems, many researchers have also discussed several concepts of the iterative methods such as Young (1971), Hackbusch (1995) and Saad (1996). In addition to these iterative methods, Abdullah (1991) initiated Half-Sweep iteration which is one of the most known and widely used iterative techniques to solve any linear systems. Differently from the Half-Sweep iteration approach, Othman and Abdullah (2000) have expanded this approach to initiate the Modified Explicit Group (MEG) method based on the quarter-sweep approach. In this study, we propose the QSAOR iterative method for solving time-fractional fractional diffusion equations based on the Caputo's implicit finite difference approximation equation. To demonstrate the capability of the QSAOR method, we also implement the Full-Sweep AOR (FSAOR) and Half-Sweep AOR (HSAOR) iterative methods being used as a control method.

Before discretizing Problem (1), let us consider some definitions that can be applied for fractional derivative theories in order to construct the approximation equation of Problem (1).

Definition 1: The Riemann-Liouville fractional integral operator, J^{α} of order-α is defined as Podlubny (1999b), Kilbas *et al*. (2006) and Zhang (2009):

$$J^{\alpha}f(x) = \frac{1}{\Gamma(a)}\int_0^x (x-t)^a f(t)dt,\ a>0,\ x>0 \quad (2)$$

Definition 2: The Caputo's fractional partial derivative operator, D^{α} of order-α is defined as Kilbas *et al*. (2006) and Caputo (1967):

$$D^{\alpha}f(x) = \frac{1}{\Gamma(m-\alpha)}\int_0^x \frac{f^{(m)}(t)}{(x-t)^{\alpha-m+1}}dt,\ a>0 \quad (3)$$

Corresponding Author: Baharuddin, State University of Medan, Medan, Indonesia

with m-1<α≤m, m∈N, X>0. From Definitions 1 and 2, Γ (α) is known as a gamma function which is given by:

$$\Gamma(\alpha) = \int_0^\infty x^{t-1}e^{-x}dx$$

For obtain the numerical solution of Time-Fractional Diffusion Equation (TFDE's) in Eq. 1 we get numerical approximation equations by using the Caputo's derivative definition with Dirichlet boundary conditions and consider the non-local fractional derivative operator. This approximation equation can be categorized as unconditionally stable scheme. On strength of Problem (1), the solution domain of the problem has been restricted to the finite space domain $0\le x\le\gamma$ with $0<\alpha<1$ whereas the parameter α refers to the fractional order of space derivative. In order to solve Problem (Eq. 1), let us consider boundary conditions of Problem (Eq. 1) be given as:

$$U(0, t) = g_0(t),\ U(l, t) = g_1(t)$$

and the initial condition:

$$U(x, 0) = f(x)$$

where, $g_0(t)$, $g_1(t)$ and $f(x)$ are given functions. Based on a discretize approximation to the time fractional derivative in Eq. 1, we consider Caputo's fractional partial derivative of order α, defined by Hadjidimos (1978) and Saad (1996):

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \frac{1}{\Gamma(n-1)}\int_0^\infty \frac{\partial u(x-s)}{\partial t}(t-s)^{-\alpha}ds,\ t>0,\ 0<\alpha<1 \tag{4}$$

Quarter-sweep implicit finite difference approximation: By using Eq. 4, the formulation of Caputo's fractional partial derivative of the first order approximation method is given as:

$$D_t^\alpha U_{i,n} \cong \sigma_{\alpha,k}\sum_{j=1}^{n}\omega_j^{(\alpha)}(U_{i,n-j+1} - U_{i,n-j}) \tag{5}$$

and we have the following expressions:

$$\sigma_{\alpha,k} = \frac{1}{\Gamma(1-\alpha)(1-\alpha)k^\alpha}$$

And:

$$\omega_j^{(\alpha)} = j^{1-\alpha} - (j-1)^{1-\alpha}$$

Previously to discretize Problem (1), let the solution domain of the problem be partitioned uniformly. To do this we consider some positive integers m and n in which the grid sizes in space and time directions for the finite difference algorithm are defined as $h = \Delta x = \gamma\text{-}0/m$ and $k = \Delta t = T/n$, respectively. Based on these grid sizes, we develop the uniformly grid network of the solution domain where the grid points in the space interval [0, γ] are shown as the numbers $x_i = ih$, $i = 0, 1, 2, ..., m$ and the grid points in the time interval [0, T] are labeled $t_j = jk$, $j = 0, 1, 2, ..., n$. Then, the values of the function U(x, t) at the grid points are denoted as $U_{i,j} = U(x_i, t_j)$. According to Eq. 5 and using the implicit finite difference discretization scheme, the quarter-sweep implicit finite difference approximation equation of Problem in Eq. 1 to the grid point centered at $(x_i, t_j) = (ih, jk)$ is given as:

$$\sigma_{\alpha,k}\sum_{j=1}^{n}\omega_j^{(\alpha)}(U_{i,n-j+1} - U_{i,n-j}) = \alpha_i\frac{1}{16h^2}(U_{i-4,n} - 2U_{i,n} + U_{i+4,n}) + b_i\frac{1}{8h}(U_{i+4,n} - U_{i-4,n}) + c_iU_{i,n} \tag{6}$$

For i = 4, 8, ..., m-4. Based on Eq. 6 this approximation equation is known as the fully implicit finite difference approximation equation which is consistent first order accuracy in time and second order in space. Particularly, the approximation Eq. 6 can be rewritten based on the specified time level. For instantce, we have for n≥2:

$$\sigma_{\alpha,k}\sum_{j=1}^{n}\omega_j^{(\alpha)}(U_{i,n-j+1} - U_{i,n-j}) = p_iU_{i-4,n} + q_iU_{i,n} + q_iU_{i,n} + r_iU_{i+4,n} \tag{7}$$

Where:

$$p_i = \frac{\alpha_i}{16h^2} = \frac{b_i}{8h}$$

$$q_i = c_i = \frac{\alpha_i}{8h^2}$$

$$r_i = \frac{\alpha_i}{16h^2} + \frac{b_i}{8h}$$

Also, we get for n = 1:

$$-p_iU_{i-4,1} + q_i^*U_{i,1} - r_iU_{i+4,i} = f_{i,1}\quad i = 4, 8, ..., m-4 \tag{8}$$

Where:

$$\omega_j^{(\alpha)} = 1$$

$$q_i^* = \sigma_{\alpha,k} - q_i$$

$$f_{i,1} = \sigma_{\alpha,k} - U_{i,1}$$

Furthermore, by using Eq. 8, it can be seen that the tridiagonal linear system can be constructed in matrix form as:

$$AU = f \tag{9}$$

Where:

$$A = \begin{bmatrix} q_4^* & -r_4 & & & & \\ -p_s & q_s^* & -r_s & & & \\ & -p_{12} & q_{12}^* & -r_{12} & & \\ & & \ddots & \ddots & \ddots & \\ & & & -p_{m-8} & q_{m-8}^* & -r_{m-8} \\ & & & & -p_{m-4} & q_{m-4}^* \end{bmatrix}_{\left(\left(\frac{m}{4}\right)-1\right)\times\left(\left(\frac{m}{4}\right)-1\right)}$$

$$U = \begin{bmatrix} U_{4,1} & U_{8,1} & U_{12,1} & L & U_{m-8,1} & U_{m-4,1} \end{bmatrix}^T$$

$$f = \begin{bmatrix} U_{4,1} + p_1 U_{01} & U_{8,1} & U_{12,1} & L & U_{m-8,1} & U_{m-4,1} + P_{m-4} U_{m,1} \end{bmatrix}^T$$

MATERIALS AND METHODS

Formulation of QSAOR method: Refer to the linear system (Eq. 9), it can be seen that the main characteristics of the coefficient matrix of the linear system have large scale and sparse. It means that the iterative methods are suitable option to solve the linear system (Young, 1971). Therefore, in this study, we consider the application of QSAOR method as linear solver (Eq. 9). Particularly, the QSAOR method is essentially the extension of the Half-Sweep Accelerated Over Relaxation (HSAOR) iterative method. The main purpose of the Quarter-Sweep iteration is to reduce the computational complexities during iteration process. Actually, development of the QSAOR method is the combination between the Quarter-Sweep iteration concept and Accelerated Over Relaxation. To derive the formulation of AOR method, let the linear system (Eq. 9) be expressed as summation of the three matrices:

$$A = D - L - V \tag{10}$$

where, D, L and V are diagonal, lower triangular and upper triangular matrices, respectively. According to Eq. 10, the QSAOR iterative method can be defined generally as (Sunarto *et al*., 2014a, b; Hadjidimos, 1978; Tian, 2003):

$$\begin{aligned} U^{(k+1)} &= (D - \omega L)^{-1}[\beta V + (\beta - \omega)D + \\ &(1 - \beta)D]U^{(k)} + \beta(D - \omega L)^{-1} f \end{aligned} \tag{11}$$

where, $U^{(k)}$ represents an unknown vector at kth iteration. Also, the implementation of the QSAOR iterative method may be described in Algorithm 1.

Algorithm 1 (QSAOR method):

i. Initializing all the parameters. Set $k = 0$.

ii. For $j = 1, 2, ..., n-1, n$ and $i = 4, 8, ..., m-8, m-4$ calculate:

$$U^{(k+1)} = (D - \omega L)^{-1}[\beta V + (\beta - \omega)D + (1 - \beta)D]U^{(k)} + \beta(D - \omega L)^{-1} f$$

iii. Convergence test. If the convergence criterion, i.e.: $\left\| \tilde{U}^{(k+1)} \quad \tilde{U}^{(k)} \right\| \leq \varepsilon = 10^{-10}$ is satisfied, go to Step (iv). Otherwise go back to Step (ii).

iv. Display approximate solutions.

RESULTS AND DISCUSSION

Numerical experiment: For the comparison purpose, two examples of time-fractional diffusion equations were considered. Both examples will be chosen from well-posed equations. Also, three different proposed iterative methods such as FSAOR, HSAOR and QSAOR will be implemented. In this study, we will consider different values of $\alpha = 0.25$, 0.50 and 0.75. For implementation of these three iterative schemes, the convergence test considered the tolerance error which is fixed as $\varepsilon = 10^{-10}$.

Examples 1: Consider the following time fractional initial boundary value problem be given as Ali *et al*. (2013):

$$\frac{\partial^\alpha U(x,t)}{\partial t^\alpha} = \frac{\partial^2 U(x,t)}{\partial t^2},\ 0 < \alpha \leq 1,\ 0 \leq x \leq \lambda,\ t > 0 \tag{12}$$

where, the boundary conditions are given in fractional terms:

$$U(0,t) = \frac{2kt^\alpha}{\Gamma(a+1)},\ U(1,t) = 1^2 + \frac{2kt^\alpha}{\Gamma(a+1)} \tag{13}$$

and the initial condition:

$$U(x, 0) = x^2 \tag{14}$$

From Problem (12) as taking $\alpha = 1$, it can be seen that Problem (12) can be reduced to the standard diffusion equation:

$$\frac{\partial U(x, t)}{\partial t} = \frac{\partial^2 U(x, t)}{\partial x^2},\ 0 \leq x \leq \gamma,\ t>0 \tag{15}$$

with the initial and boundary conditions:

$$U(x, 0) = x^2,\ U(0, t) = 2kt,\ U(1, t) = 1^2 + 2kt$$

Then, the analytical solution of Problem (14) is obtained as follows:

$$U(x,t) = x^2 + 2kt$$

Now by applying the series:

Table 1: Comparison between No. of iterations (K), the execution time (sec) and maximum errors for the iterative methods using examples 1 at $\alpha = 0.25$, 0.50, 0.75

		$\alpha = 0.25$			$\alpha = 0.50$			$\alpha = 0.75$		
M	Methods	K	Time	Max. error	K	Time	Max. error	K	Time	Max. error
128	FSAOR	657	1.93	9.95e-05	615	1.83	9.84e-05	613	1.79	1.29e-04
	HSAOR	517	1.78	9.95e-05	510	1.54	9.84e-05	599	1.39	1.29e-04
	QSAOR	234	0.91	9.9e-05	396	1.02	9.84e-05	337	1.03	1.29e-04
256	FSAOR	870	6.38	9.95e-05	659	2.99	9.84e-05	630	1.72	1.29e-04
	HSAOR	857	3.02	9.95e-05	425	2.93	9.84e-05	413	1.50	1.29e-04
	QSAOR	255	1.11	9.95e-05	215	1.36	9.84e-05	212	1.33	1.29e-04
512	FSSOR	4940	31.35	9.96e-05	3080	22.62	9.84e-05	1301	16.67	1.29e-04
	FSAOR	870	6.54	9.05-e05	859	6.37	9.84e-05	830	6.45	1.29e-04
	QSAOR	452	2.31	9.95e-05	417	2.23	9.84e-05	448	2.27	1.29e-04
1024	FSSOR	19033	45.72	9.97e-05	12232	101.21	9.80-e05	5911	105.27	1.30e-04
	HSAOR	4940	33.28	9.96e-05	3080	42.68	9.80e-05	1301	17.87	1.29e-04
	QSAOR	2673	20.97	9.96e-05	1664	13.26	9.84e-05	698	5.95	1.29e-04
2048	FSSOR	70547	1142.09	1.00e-04	45700	1342.00	9.89e-05	22474	1002.85	1.30e-04
	HSAOR	19033	512.59	1.00e-04	12232	338.14	9.85e-05	5911	157.02	1.30e-04
	QSAOR	10332	158.37	9.96e-05	6633	100.86	9.85e-05	3200	49.16	1.30e-04

Table 2: Comparison between number of iterations (K), the execution time (sec.) and maximum errors for the iterative methods using examples 2 at $\alpha = 0.25$, 0.50, 0.75

		$\alpha = 0.25$			$\alpha = 0.50$			$\alpha = 0.75$		
M	Methods	K	Time	Max. error	K	Time	Max. error	K	Time	Max. error
128	FSAOR	370	6.19	1.94e-02	260	5.86	8.29e-02	257	5.80	1.37e-01
	HSAOR	199	5.11	1.95e-02	154	5.04	8.30e-02	146	5.02	1.37e-01
	QSAOR	72	2.79	1.95e-02	61	2.05	8.30e-02	48	2.25	1.37e-01
256	FSAOR	1761	19.30	1.95e-02	809	11.82	8.29e-02	291	7.39	1.37e-01
	HSAOR	545	6.25	1.95e-02	328	5.70	8.29e-02	163	5.29	1.37e-01
	QSAOR	166	3.02	1.95e-02	116	2.39	8.30e-02	72	2.40	1.37e-01
512	FSSOR	6746	125.09	1.94e-02	3240	63.93	8.29e-02	1471	30.79	1.37e-01
	FSAOR	2246	19.45	1.95e-02	1402	14.27	8.29e-02	625	9.09	1.37e-01
	QSAOR	849	7.63	1.95e-02	629	6.97	8.30e-02	279	4.77	1.37e-01
1024	FSSOR	25054	866.80	1.94e-02	12126	435.44	8.29e-02	5644	208.20	1.37e-01
	HSAOR	8478	117.00	1.94e-02	5313	76.75	8.29e-02	2461	38.54	1.37e-01
	QSAOR	3301	28.86	1.95e-02	2425	23.05	8.29e-02	1099	13.21	1.37e-01
2048	FSSOR	91984	6092.40	1.94e-02	44563	1428.60	8.29e-02	20921	1609.51	1.37e-01
	HSAOR	40541	1109.00	1.94e-02	19650	557.72	8.29e-02	9198	268.45	1.37e-01
	QSAOR	18209	272.76	1.94e-02	8832	139.41	8.29e-02	4126	68.64	1.37e-01\

$$U(x,t) = \sum_{n=0}^{m-1} \frac{\partial^n U(x,0)}{\partial t^n}\frac{t^n}{n!} + \sum_{n=1}^{\infty}\sum_{i=0}^{m-1} \frac{\partial^{mn+i}U(x,0)}{\partial t^{mn+i}}\frac{t^{n\alpha+i}}{\Gamma(n\alpha+i+1)}$$

To U(x, t) for $0<\alpha\leq 1$ it can be shown that the analytical solution of Problem (16) is given as:

$$U(x,\, t) = x^2 + 2k\frac{t^{\alpha}}{\Gamma(a+1)}$$

Examples 2: Let us consider the following time fractional initial boundary value problem be defined as Ali *et al.* (2013):

$$\frac{\partial^{\alpha} U(x,\, t)}{\partial t^{a}} = \frac{1}{2}x^2\frac{\partial^2 U(x,\, t)}{\partial x^2},\, 0 < a \leq 1,\, 0 \leq x \leq \gamma,\, t > 0 \quad (16)$$

where the boundary conditions are given in fractional terms:

$$U(0,\, t) = 0,\, U(1,\, t) = e^t \quad (17)$$

and the initial condition:

$$U(x,\, 0) = x^2 \quad (18)$$

From Problem (19) as taking $\alpha = 1$, it can be shown that Eq. 19 can also be reduced to the standard diffusion equation:

$$\frac{\partial U(x,\, t)}{\partial t} = \frac{1}{2}x^2\frac{\partial^2 U(x,\, t)}{\partial x^2},\, 0 \leq x \leq \gamma,\, t > 0 \quad (19)$$

Then, the analytical solution of Problem (17) is obtained as follows:

$$U(x,\, t) = x^2 e^t$$

Now by applying the series:

$$U(x,\, t) = \sum_{n=0}^{m-1} \frac{\partial^n U(x,\, 0)}{\partial t^n}\frac{t^n}{n!} + \sum_{n=1}^{\infty}\sum_{i=0}^{m-1} \frac{\partial^{mn+i}U(x,\, 0)}{\partial t^{mn+i}}\frac{t^{n\alpha+i}}{\Gamma(n\alpha+i+1)}$$

to U(x, t) for $0<\alpha\leq 1$ it can be shown that the analytical solution of Problem (17) is stated as (Table 1 and 2):

$$U(x,t) = x^2 \left[1 + \frac{t^{\alpha}}{\Gamma(\alpha+1)} + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + ... \right]$$

All results of numerical experiments for Problems (12) and (15), obtained from implementation of FSAOR, HSAOR and QSAOR iterative methods are recorded in Table 1 and 2 at different values of mesh sizes, M = 128, 256, 512, 1024 and 2048.

CONCLUSION

As a conclusion for the numerical solution of the time-fractional diffusion problems this study deals with the implementation of QSAOR iterative method to solve a linear system generated by the Quarter-Sweep Caputo's implicit approximation equations. Through numerical experiments results from Table 1 and 2, clearly it demonstrates that two promising improvements in the number of iteration (K) and execution time with implementing a QSAOR iterative method have been shown as compared to the FSAOR and HSAOR methods. Overall, the numerical results showed that the quarter-sweep iteration concept in association with the AOR iterative method is superior and it has reduced the computational complexity significantly. Therefore further investigation of two-step (Evans and Sahimi, 1988; Ruggiero and Galligani, 1990; Sahimi *et al.*, 1993) iterative methods should be considered for future works.

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