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Implicit finite difference solution for time-fractional diffusion equations using AOR method

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Abstract. In this paper, we derive an implicit finite difference approximation equation of the one-dimensional linear time fractional diffusion equations, based on the Caputo's time fractional derivative. Then this approximation equation leads the corresponding system of linear equation, which is large scale and sparse. Due to the characteristics of the coefficient matrix, we use the Accelerated Over-Relaxation (AOR) iterative method for solving the generated linear system. One example of the problem is presented to illustrate the effectiveness of AOR method. The numerical results of this study show that the proposed iterative method is superior compared with the existing one weighted parameter iterative method.

1. Introduction

Fractional partial derivatives equations (FPDEs) have been used to govern many linear problems in variety of applications field by using fractional calculus. Therefore, there are many successful mathematical models, which are based on fractional partial derivative equations (FPDEs), have been developed [1,2,3,4]. Supposing a fractional derivative replaces the first-order time partial derivative in a diffusion model, in which this matter leads to slower diffusion [1]. In order to solve one dimensional time fractional diffusion model with constant coefficient, analytical and/or numerical solutions are available by using transform method which is one method used for numerical solution of the fractional diffusion equations (FDE) [1,5,6], finite elements together with the methods of line [3], explicit and implicit finite difference methods [7,8,9]. In fact, these finite difference schemes are available in the literature [9,10]. As we know, the explicit methods are conditionally stable.

Due to the stability of finite difference discretization schemes, this paper deals with the application of the implicit discretization scheme and Caputo's fractional partial derivative of order α being used to discretize the time-fractional diffusion equation and derive a Caputo's implicit finite difference approximation equation. This implicit finite difference approximation equation will lead the tridiagonal linear system in which the properties of the coefficient matrix of the linear system are sparse and large scale. To solve any linear system, it can be observed that many iterative methods have been proposed to solve any system of linear equations. For instance, further discussions of various iterative methods have been discussed by Young [12, 13, 14], Hackbusch [15] and Saad [16]. In addition to that, Evans [17] has also proposed block iterative methods via the Explicit Group (EG) iterative methods in order to speed up the convergence rate. Again, it can be found that the concept of block iteration has been expanded by other researchers (Ibrahim and Abdullah [23], Evans and Yousif

[24, 25], Yousif and Evans [18]) to demonstrate the efficiency of its computation cost. Apart from these iterative methods, Hadjidimos [20] has initiated the Accelerated Over-Relaxation (AOR) method which is based on the point iterative method together with two weighted parameters. Actually this iterative method is one of the efficient point iterative methods. To accelerate the convergence rate of this iteration, Martins *et al* [26] have combined the concept of the EG iterative method together with the AOR method in which this combination is called as the Explicit Group AOR (EGAOR) method for solving elliptic partial differential equations. They pointed out that the 4 point-EGAOR method is superior compared to the existing point AOR method.

Due to the advantages of the AOR method for solving partial differential equations, the main objective of this paper is to construct and examine the effectiveness of the Accelerated Over-Relaxation (AOR) iterative method for solving time fractional parabolic partial differential equations (TPPDE's) based on the Caputo's implicit finite difference approximation equation. To examine the effectiveness of the AOR method, we also implement the Successive Over-Relaxation (SOR) and Gauss-Seidel (GS) iterative methods being used control methods.

To indicate the effectiveness of AOR method, let time fractional parabolic partial differential equation (TPPDE's be defined as

$$\frac{\partial^\alpha U(x,t)}{\partial^\alpha} = a(x)\frac{\partial^2 U(x,t)}{\partial x^2} + b(x)\frac{\partial U(x,t)}{\partial x} + c(x)U(x,t) \tag{1}$$

where *a(x), b(x)* and *c(x)* are known functions or constants, whereas α is a parameter which refers to the fractional order of time derivative.

2. Preliminaries

Before constructing the finite difference approximation equation of Problem (1), we need to consider the following definitions and properties of fractional derivative theory which are used in this paper.

Definition 1. [11] The Riemann-Liouville fractional integral operator , J^α of order-α is defined as

$$J^\alpha f(x) = \frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^\alpha f(t)dt\,,\ \ \alpha > 0\,,\ x > 0 \tag{2}$$

Definition 2.[11] The Caputo's fractional partial derivative operator, D^α of order -α is defined as

$$D^\alpha f(x) = \frac{1}{\Gamma(m-\alpha)}\int_0^x \frac{f^{(m)}(t)}{(x-t)^{\alpha-m+1}}dt, \quad \alpha > 0 \tag{3}$$

with $m-1 < \alpha \le m, \mathrm{m} \in \mathrm{N},\ x > 0$

To find the numerical solution of Problem (1), we derive the Caputo's implicit finite difference approximations of Problem (1) with Dirichlet boundary conditions. To do this, we need to consider the non-local fractional derivative operator. This approximation equation can be categories as unconditionally stable scheme .Based on Problem (1), the solution domain of the problem has been restricted to the finite space domain $0 \le x \le \gamma$, with $0 < \alpha < 1$, whereas the parameter α refers to the fractional order of time derivative. To solve Problem (1), let us assume the initial and boundary conditions of Problem (1) be given as

$$U(0,t) = g_0(t),\ U(\ell,t) = g_1(t),$$

and the initial condition

$$U(x,0) = f(x),$$

where $g_0(t)$, $g_1(t)$, and $f(x)$, are given functions. A discretize approximation to the time fractional derivative in Eq. (1), we consider Caputo's fractional partial derivative of order α, defined by [11,13,14],

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \frac{1}{\Gamma(n-1)}\int_0^\infty \frac{\partial u(x-s)}{\partial t}(t-s)^{-\alpha}\,ds, \quad t>0, \quad 0<\alpha<1 \tag{4}$$

The organization of the paper is as follows: In Section 2, the formula of the Caputo's fractional derivative operator and numerical procedure for solving time fractional diffusion equation (1) by means of the implicit finite difference method are given. In Section 3, formulation of the AOR iterative method is introduced. In Section 4 shows numerical example and its results and conclusion is given in Section 5.

3. Approximation For Fractional Diffusion Equation

In this section, the first order approximation method for the computation of Caputo's fractional partial derivative is then stated as the following expression

$$D_t^\alpha U_{i,n} \cong \sigma_{\alpha,k}\sum_{j=1}^{n}\omega_j^{(\alpha)}\left(U_{i,n-j+1}-U_{i,n-j}\right) \tag{5}$$

In which we define

$$\sigma_{\alpha,k} = \frac{1}{\Gamma(1-\alpha)(1-\alpha)k^\alpha} \quad \text{and} \quad \omega_j^{(\alpha)} = j^{1-\alpha}-(j-1)^{1-\alpha}.$$

By using formula (5), we attempt to derive a discretize equation of Problem (1). Before constructing the discretize equation, let the solution domain of the problem need to be partitioned uniformly. For some positive integers *m* and *n*, the grids sizes in space and time directions for the finite difference algorithm are defined as $h=\Delta x=\frac{\gamma-0}{m}$ and $k=\Delta t=\frac{T}{n}$ respectively. To construct the uniformly grid network of the solution domain, let the grid points in the space interval $[0,\gamma]$ be indicated as the numbers $x_i = ih,\ i=0,1,2,...,m$ and the grid points in the time interval $[0,T]$ are labeled $t_j = jk,\ j=0,1,2,...,n$. The values of the function $U(x,t)$ at the grid points are denoted as $U_{i,j}=U(x_i,t_j)$.

Using Eq. (5) and the implicit finite difference discretization scheme, the Caputo's discrete equation of Problem (1) to the grid point centered at $(x_i,t_j)=(ih,nk)$ is given as

$$\begin{aligned}&\sigma_{\alpha,k}\sum_{j=1}^{n}\omega_j^{(\alpha)}\left(U_{i,n-j+1}-U_{i,n-j}\right)\\ &= a_1\frac{1}{h^2}\left(U_{i-1,n}-2U_{i,n}+U_{i+1,n}\right)+b_i\frac{1}{2h}\left(U_{i+1,n}-U_{i-1,n}\right)+c_iU_{i,n},\end{aligned} \tag{6}$$

for *i=1,2...,m-1.*

According to Eq. (6), the discrete equation is known as the fully implicit finite difference approximation equation which is consistent first order accuracy in time and second order in space. Again the approximation equation (6) can be rewritten based on the specified time level. Therefore, we have for $n \ge 2$:

$$\sigma_{\alpha,k}\sum_{j=1}^{n}\omega^{(\alpha)}\left(U_{i,n-j+1}-U_{i,n-j}\right)=\left(\frac{a_i}{h^2}-\frac{b_i}{2h}\right)U_{i-1,n}+\left(c_i-\frac{2a_i}{h^2}\right)U_{i,n}+\left(\frac{a_i}{h^2}+\frac{b_i}{2h}\right)U_{i+1,n}, \tag{7a}$$

$$\therefore \sigma_{\alpha,k}\sum_{j=1}^{n}\omega_j^{(\alpha)}\left(U_{i,n-j+1}-U_{i,n-j}\right)=p_iU_{i-1,n}+q_iU_{i,n}+r_iU_{i+1,n},$$

where

$$p_i=\frac{a_i}{h^2}-\frac{b_i}{2h},$$

$$q_i=c_i-\frac{2a_i}{h^2},$$

$$r_i=\frac{a_i}{h^2}+\frac{b_i}{2h}.$$

Also, we get for $n = 1$,

$$-p_iU_{i-1,1}+q_i^*U_{i,1}-r_iU_{i+1,1}=f_{i,1},\quad i=1,2,...,m-1 \tag{7b}$$

where

$$\omega_j^{(\alpha)}=1,$$

$$q_i^*=\sigma_{\alpha,k}-q_i,$$

$$f_{i,1}=\sigma_{\alpha,k}U_{i,1}.$$

Based on Eq. (7b), it can be seen that the tridiagonal linear system can be constructed in a matrix form as

$$A\underset{\sim}{U}=\underset{\sim}{f} \tag{8}$$

where

$$A=\begin{bmatrix} q* & -r & & & & \\ -p & q* & -r & & & \\ & -p & q* & -r & & \\ & & \ddots & \ddots & \ddots & \\ & & & -p & q* & -r \\ & & & & -p & q* \end{bmatrix}_{(m-1)x(m-1)},$$

$$\underset{\sim}{U}=\begin{bmatrix}U_{11} & U_{21} & U_{31} & \cdots & U_{m-2,1} & U_{m-1,1}\end{bmatrix}^T,$$

$$\underset{\sim}{f}=\begin{bmatrix}U_{11}+p_1U_{01} & U_{21} & U_{31} & \cdots & U_{m-2,1} & U_{m-1,1}+p_{m-1}U_{m,1}\end{bmatrix}^T.$$

4. Formulation of Accelerated Over-Relaxation Iterative Method

Based on the tridiagonal linear system in Eq. (8), it is clear that the characteristics of its coefficient matrix are large scale and sparse. As mentioned in Section 1, many researchers have discussed various iterative methods such as Young [12,13,14], Hackbusch [15], Saad [16], Evans [17], Yousif and Evans [18,19], and Othman and Abdullah [20]. To obtain numerical solutions of the tridiagonal linear system which is generated by the Caputo's implicit finite difference approximation equation, we consider the Accelerated Over-Relaxation (AOR) iterative method [20, 21], which is the most known and widely using for solving any linear systems. To formulate AOR method, let the coefficient matrix A in (8) be expressed as summation of the three matrices

$$A=D-L-V \tag{9}$$

where D, L and V are diagonal, lower triangular and upper triangular matrices respectively.

Thus, AOR iterative method can be defined generally as [21, 22]

$$\widetilde{U}^{(k+1)} = (D-\omega L)^{-1}\left[\beta V + (\beta-\omega)D + (1-\beta)D\right]\widetilde{U}^{(k)} + \beta(D-\omega L)^{-1} f \tag{10}$$

where $\widetilde{U}^{(k)}$ represents an unknown vector at k^{th} iteration. The implementation of the AOR iterative method can be described in Algorithm 1.

Algorithm 1: AOR method

i. Initialize $\widetilde{U} \leftarrow 0$ and $\varepsilon \leftarrow 10^{-10}$.

ii. For $j = 1,2,\ldots,n$ Implement

 a. For $i = 1,2,\ldots,m-1$ calculate

$$\widetilde{U}^{(k+1)} = (D-\omega L)^{-1}\left[\beta V + (\beta-\omega)D + (1-\beta)D\right]\widetilde{U}^{(k)} + \beta(D-\omega L)^{-1} f$$

 b. Convergence test. If the convergence criterion i.e $\left\|\widetilde{U}^{(k+1)} - \widetilde{U}^{(k)}\right\| \le \varepsilon = 10^{-10}$ is satisfied, go to Step (iii). Otherwise go back to Step (a).

iii Display approximate solutions.

5. Numerical Example

In this section, we apply the approximation equation (7), in order in getting numerical solution of time fractional diffusion equation. One example of the time fractional diffusion equation was tested for verify the effectiveness of the Gauss-Seidel (GS), Successive Over-Relaxation (SOR) and Accelerated Over-Relaxation (AOR) iterative methods. For comparison purpose in indicating the effectiveness of these three proposed iterative methods, three criteria will be considered such as number of iterations, execution time (in seconds) and maximum absolute error at three different values of $\alpha = 0.25$, $\alpha = 0.50$ and $\alpha = 0.75$. For implementation of point iterations, the convergence test considered the tolerance error, which is fixed as $\varepsilon = 10^{-10}$.

Consider the time fractional initial boundary value problem be given as [22]

$$\frac{\partial^{\alpha} U(x,t)}{\partial t^{\alpha}} = \frac{\partial^{2} U(x,t)}{\partial x^{2}}, \quad 0 < \alpha \le 1, 0 \le x \le \gamma, \quad t > 0, \tag{11}$$

where the boundary conditions are stated in fractional terms

$$U(0,t) = \frac{2kt^{\alpha}}{\Gamma(\alpha+1)}, \; U(\ell,t) = \ell^{2} + \frac{2kt^{\alpha}}{\Gamma(\alpha+1)}, \tag{12}$$

and the initial condition

$$U(x,0) = x^{2}. \tag{13}$$

From Problem (11), as taking $\alpha = 1$, it can be seen that Eq. (11) can be reduced to the standard diffusion equation

$$\frac{\partial U(x,t)}{\partial t} = \frac{\partial^{2} U(x,t)}{\partial x^{2}}, \quad 0 \le x \le \gamma, \quad t > 0, \tag{14}$$

subjected to the initial condition

$$U(x,0) = x^{2},$$

and boundary conditions

$$U(0,t) = 2kt, \qquad U(\ell,t) = \ell^{2} + 2kt,$$

Then the analytical solution of Problem (14) is obtained as follows

$$U(x,t) = x^2 + 2kt.$$

Now by applying the series

$$U(x,t) = \sum_{n=0}^{m-1} \frac{\partial^n U(x,0)}{\partial t^n} \frac{t^n}{n!} + \sum_{n=1}^{\infty} \sum_{i=0}^{m-1} \frac{\partial^{mn+i} U(x,0)}{\partial t^{mn+i}} \frac{t^{n\alpha+i}}{\Gamma(n\alpha+i+1)}$$

to $U(x,t)$ for $0 < \alpha \leq 1,$ it can be shown that the analytical solution of Problem (11) is given as

$$U(x,t) = x^2 + 2k\frac{t^\alpha}{\Gamma(\alpha+1)}.$$

All results of numerical experiments for Problem (11), obtained from implementation of GS, SOR and AOR iterative methods are recorded in Table 1 at different values of mesh sizes, m = 128, 256, 512, 1024, and 2048.

TABLE 1. Comparison of number iterations, the execution time (seconds) and maximum errors for the iterative methods using example at $\alpha = 0.25, 0.50, 0.75$

M	Method	α = 0.25			α = 0.50			α = 0.75		
		K	Time	Max Error	K	Time	Max Error	K	Time	Max Error
128	**GS**	21017	37.73	9.97e-5	13601	5.92	9.86e-5	6695	2.94	1.30e-4
	SOR	465	1.54	9.95e-5	385	1.44	9.84e-5	411	1.51	1.30e-4
	AOR	431	0.33	9.95e-5	185	0.33	9.84e-5	411	0.69	1.29e-4
256	**GS**	77231	343.63	1.00e-4	50095	42.17	9.90e-5	24732	20.70	1.30e-4
	SOR	939	4.73	9.96e-5	769	3.99	9.84e-5	513	2.92	1.30e-4
	AOR	870	1.25	9.95e-5	709	2.27	9.84e-5	513	1.48	1.29e-4
512	**GS**	281598	2747.34	1.02e-4	183181	339.85	1.01e-4	90783	166.75	1.32e-4
	SOR	1886	17.51	9.96e-5	1537	14.39	9.84e-5	1025	10.33	1.30e-4
	AOR	1607	8.16	9.95e-5	1481	9.91	9.82e-5	1025	5.79	1.29e-4
1024	**GS**	1017140	68285.36	1.09e-4	663971	2454.53	1.08e-5	330622	1209.39	1.40e-4
	SOR	3805	68.91	9.96e-5	3073	55.68	9.84e-5	2049	37.33	1.30e-4
	AOR	3100	28.30	9.95e-5	2918	14.39	9.84e-5	2049	19.36	1.29e-4
2048	**GS**	3631638	58914.30	1.38e-4	2380946	17795.25	1.38e-4	1192528	8794.26	1.71e-4
	SOR	8193	430.17	9.96e-5	6145	239.84	9.84e-5	7849	303.50	1.30e-4
	AOR	8139	144.51	9.95e-5	2819	27.23	9.83e-5	7849	135.97	1.29e-4

6. Conclusion

For the time fractional diffusion problems, the paper presents the formulation of the Caputo's implicit finite difference equations to generate a linear system. Then to solve the generated linear system, the formulation and implementation of these three proposed iterative methods such as GS, SOR and AOR have been presented based on the Caputo's implicit finite difference approximation equation. From observation of all experimental results by imposing the GS, SOR and AOR iterative methods, it is obvious at $\alpha = 0.25$ that that number of iterations declined approximately by 0.66-18.53% corresponds to the AOR iterative method compared with the SOR method. Again in terms of execution time, implementations of AOR method are much faster about 53.39-78.57% than the SOR method. It means that the AOR method requires the least amount for number of iterations and computational time at $\alpha = 0.25$ as compared with GS and SOR iterative methods. This is due to the implementations of AOR iterative method have been accelerated by using the optimal value of the two weighted parameters, ω and β. In fact, these conclusions are inline with the results of Othman and

Abdullah [19]. Based on the accuracy of there three iterative methods, it can be concluded that the numerical solutions for AOR method are in good agreement.

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