Application Half-Sweep Preconditioned SOR Method For Solving Time-Fractional Diffusion Equations

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Abstract

This research examines the performance of the application Half-Sweep Preconditioned SOR (HSPSOR) method together with an unconditionally implicit Caputo's time–fractional finite difference approximation equation for solving time-fractional partial diffusion equations (TFPDE's). To do it, the implicit Caputo's time-fractional approximation equations and preconditioned matrix are used to construct the corresponding preconditioned linear system. In addition to that, formulation and application the HSPSOR method are also presented. Based on numerical results of the proposed iterative method, it can be concluded that the proposed iterative method is superior to the Full-Sweep PSOR iterative methods.

Keywords :HSPSOR, Implicit Finite Different, Caputo'S, Time-Fractional

1.Introduction

Based on previous studies in (Dey, 1999; Diethelm & Freed, 1999; Gorenflo & Mainardi, 1997; Sene & Fall, 2019; Tam, Wei, & Jin, 2005) many model in mathematical use fractional partial differential equations (FPDEs) to solve fractional problems such as time-fractional partial diffusion equations (TFPDE's). Following to that, there are several methods used to solve these models. For instance, we have transform method (Sene & Fall, 2019), which is used to obtain analytical and/or numerical solutions of the fractional diffusion equations (FDE's). Other than this method, other researchers have proposed finite difference methods such as explicit and implicit (Chaves, 1998; Evans & Haghighlt, 1984; Kepczynska, 2005). Also it is pointed out that the explicit iterative methods are conditionally stable. Therefore, to solve the problems of the TFPDE's needs to be discretized. By using the implicit finite difference scheme and Caputo fractional operator, a linear system at each time level can be construct through the approximation equations. Therefore, Due to the matrix properties of the linear system, iterative methods are the alternative option for efficient solutions

As far as iterative methods are concerned, it can be observed that many researchers such as (Cheng, Huang, & Cheng, 2006; Hackbusch, 2016; Young, 1971) then (Saad, 2003) have proposed and discussed several families of iterative methods. Among the existing proposed methods, the preconditioned iterative methods (Gunawardena, Jain, & Snyder, 1991; Salkuyeh & Shamsi, 2012; Shen, Zong, & Shao, 2009) have been widely accepted to be one of the efficient methods for solving linear systems.

Because of the advantages of these iterative methods, the aim of this study is to develop and application the performance of the HSPSOR method to solve (TFPDE's) based on the implicit Caputos's time-fractional finite difference approximation equation. To application the performance of the HSPSOR, we also applying Full-Sweep Preconditioned SOR (FSPSOR).

To demonstrate the performance of HSPSOR, let TFPDE's be given as

$$\frac{\partial^{\alpha} Z(y,r)}{\partial^{\alpha}} = a(y)\frac{\partial^{2} Z(y,r)}{\partial y^{2}} + b(y)\frac{\partial Z(y,r)}{\partial y} + c(y)Z(y,r) \quad (1)$$

where *a(y), b(y)* and *c(y)* are clear functions or fixed, whereas the parameter α refers to the fractional order TFPDE's derivative.

2. Preliminaries

Previous to construct the discrete equation of Eq (1), in this section given some definitions can be applied for fractional derivative theory

Definition 1.(Sunarto, 2014; Young, 1971) The Riemann-Liouville operator is defined as

$$J^{\alpha} f(y) = \frac{1}{\Gamma(\alpha)}\int_{0}^{x}(y-r)^{\alpha} f(r)dr \text{ , } \alpha > 0 \text{ , } y > 0 \quad (2)$$

Definition 2. (Sunarto, 2014; Young, 1971) The Caputo's operator is defined as

$$D^{\alpha} f(y) = \frac{1}{\Gamma(m-\alpha)}\int_{0}^{x}\frac{f^{(m)}(r)}{(y-r)^{\alpha-m+1}}dr, \quad \alpha > 0 \quad (3)$$

with $m-1 < \alpha \le m$, $m \in N$, $y > 0$

To obtain the numerical solution of Eq. (1) with Dirichlet boundary conditions, we get numerical approximations by using the Caputo's derivative definition and consider the non-local fractional TFPDE's derivative operator. This approximation equation can be categorized as unconditionally stable scheme. On strength of Problem (1), the solution domain of the problem has been restricted to the finite space domain $0 \le y \le \gamma$ and $0 < \alpha < 1$, whereas α refers to the fractional order of TFPDE's derivative.

In addition to that, consider Initial and boundary (conditions) of Eq (1) be given as

$Z(x,0) = f(y)$, and $Z(0,r) = g_0(r)$, $Z(l,r) = g_1(r)$, with $g_0(r)$, $g_1(r)$, and $f(y)$,

are given clear functions.

A discretize approximation to TFPDE's Eq. (1) by using Caputo's order-α, is given as [9,11]

$$\frac{\partial^{\alpha} Z(x,r)}{\partial r^{\alpha}} = \frac{1}{\Gamma(n-1)}\int_{0}^{\infty}\frac{\partial u(y-s)}{\partial r}(r-s)^{-\alpha} ds, \quad r > 0, \quad 0 < \alpha < 1 \quad (4)$$

3. Approximation For Fractional Diffusion Equations

Before constructing the Half-Sweep Caputo's implicit approximation equation, let the Caputo's fractional partial derivative in Eq.(4) be rewritten as

$$D_t^{\alpha} U_{i,n} \cong \sigma_{\alpha,k} \sum_{j=1}^{n} \omega_j^{(\alpha)} \left(U_{i,n-j+1} - U_{i,n-j}\right)$$

(5)

and we have the following expressions

$$\sigma_{\alpha,k} = \frac{1}{\Gamma(1-\alpha)(1-\alpha)k^{\alpha}}$$

and

$$\omega_j^{(\alpha)} = j^{1-\alpha} - (j-1)^{1-\alpha}.$$

To facilitate us in discretizing the fractional Eq. (1) via the implicit finite difference discretization scheme, Firstly, its the solution domain of the fractional problem is divided uniformly into several subintervals. To do this, we consider several positive integers *m* and *n* in which the sizes of subinterval over space and time directions are stated as $h = \Delta x = \frac{\gamma - 0}{m}$ and $k = \Delta t = \frac{T}{n}$ respectively. According to these sizes of subinterval, we develop the uniformly finite grid network in the solution domain where the grid points over the space interval $[0, \gamma]$ and time interval $[0, T]$ are denoted $x_i = ih,\ i = 0,1,2,...,m$ and $t_j = jk,\ j = 0,1,2,...,n$ respectively. Then approximate values of the function $U(x,t)$ at the grid points are labeled as $U_{i,j} = U(x_i, t_j)$. By considering Eq. (5) and the Half-Sweep implicit finite difference discretization scheme, the Half-Sweep Caputo's implicit approximation equation of fractional Eq (1) to the reference grid point at $(x_i, t_j) = (ih, jk)$ is can be shown as

$$\sigma_{\alpha,k} \sum_{j=1}^{n} \omega_j^{(\alpha)} \left(U_{i,n-j+1} - U_{i,n-j} \right)$$
$$= a_i \frac{1}{4h^2} \left(U_{i-2,n} - 2U_{i,n} + U_{i+2,n} \right) + b_i \frac{1}{4h} \left(U_{i+2,n} - U_{i-2,n} \right) + c_i U_{i,n}, \quad (6)$$

for $i=2,4,...,m-2$. Actually, the approximation equation (6) is also categories as one of family of fully implicit finite difference approximation equations. Also this equation is first order accuracy in time direction and second order in space direction. For simplicity, consider the Half-Sweep approximation equation (6) be rewritten at the specified time level. For instance, we have for $n \geq 2$:

$$\sigma_{\alpha,k} \sum_{j=1}^{n} \omega_j^{(\alpha)} \left(U_{i,n-j+1} - U_{i,n-j} \right) = p_i U_{i-2,n} + q_i U_{i,n} + r_i U_{i+2,n}, \quad (7)$$

where

$$p_i = \frac{a_i}{4h^2} - \frac{b_i}{4h},\ q_i = c_i - \frac{a_i}{2h^2},\ r_i = \frac{a_i}{4h^2} + \frac{b_i}{4h}.$$

Also, we get for $n = 1$,

$$-p_i U_{i-2,1} + q_i^* U_{i,1} - r_i U_{i+2,1} = f_{i,1}, \quad i = 2,4,...,m-2 \quad (8)$$

where

$$\omega_j^{(\alpha)} = 1,\ q_i^* = \sigma_{\alpha,k} - q_i,\ f_{i,1} = \sigma_{\alpha,k} U_{i,1}.$$

Referring to Eq. (8), we can get the tridiagonal linear system that can be constructed in matrix form as

$$A\underset{\sim}{U} = \underset{\sim}{f} \quad (9)$$

where

$$A = \begin{bmatrix} q_2^* & -r_2 & & & & \\ -p_4 & q_4^* & -r_4 & & & \\ & -p_6 & q_6^* & -r_6 & & \\ & & \ddots & \ddots & \ddots & \\ & & & -p_{m-2} & q_{m-2}^* & -r_{m-2} \\ & & & & -p_{m-2} & q_{m-2}^* \end{bmatrix}_{\left(\left(\frac{m}{2}\right)-1\right) x \left(\left(\frac{m}{2}\right)-1\right)},$$

$$\underset{\sim}{U} = \left[U_{2,1} \quad U_{4,1} \quad U_{6,1} \quad \ldots \quad U_{m-4,1} \quad U_{m-2,1} \right]^T, \underset{\sim}{f} = \left[U_{21} + p_1 U_{01} \quad U_{41} \quad U_{61} \quad \ldots \quad U_{m-4,1} \quad U_{m-2,1} + p_{m-2} U_{m,1} \right]^T$$

4. Analysis Of Stability

In this section, we have considered the stability analysis of the implicit finite difference approximation equation in Eq.(1). For stability analysis, we will use Von-Neumann's theorem (Langlands & Henry, 2005) and the Lax equivalence theorem (I., Richtmyer, & Morton, 1968). It follows that the numerical solution of the approximation equation in Eq.(1) converges to the exact solution as $h, k \to 0$.

Theorem 4.1.
The fully implicit numerical method Eq.(1), the solution to Eq.(1) with $0 < \alpha < 1$ *on the finite domain* $0 \le x \le 1$*, with zero boundary condition* $U(0,t) = U(1,t) = 0$ *for all* $t \ge 0$*, is consistent and unconditionally stable.*

Proof. To examine the stability of the proposed method, we findfor solution of the form $U_j^n = \xi_n e^{i\omega jh}$, $i = \sqrt{-1}$, ω real. Therefore Eq. (1) becomes

$$\sigma_{\alpha,k}\xi_{n-1}e^{i\omega jh} - \sigma_{\alpha,k}\sum_{j=2}^{n}\omega_j^{(\alpha)}\left(\xi_{n-j+1}e^{i\omega jh} - \xi_{n-j}e^{i\omega jh}\right) = -p_i\xi_n e^{i\omega(j-4)h} + \left(\sigma_{\alpha,k} - q_i\right)\xi_n e^{i\omega jh} - r_i\xi_n e^{i\omega(j+4)h} \tag{10}$$

by simplifying and reordering over Eq.(10), we have :

$$\sigma_{\alpha,k}\xi_{n-1} - \sigma_{\alpha,k}\sum_{j=2}^{n}\omega_j^{(\alpha)}\left(\xi_{n-j+1} - \xi_{n-j}\right) = \xi_n\left(\left(\left(-p_i - r_i\right)\cos(\omega h)\right) + \left(\sigma_{\alpha,k} - q_i\right)\right).$$

this can be reduced to :

$$\xi_n = \frac{\xi_{n-1} + \sum_{j=2}^{n}\omega_j^{(\alpha)}\left(\xi_{n-j} - \xi_{n-j+1}\right)}{\left(1 + \frac{(p_i + r_i)}{\sigma_{\alpha,k}}\cos(\omega h) + \frac{q_i}{\sigma_{\alpha,k}}\right)} \tag{11}$$

From Eq.(11), it can be observed that the conducted as

$$\left(1 + \frac{(-p_i - r_i)}{\sigma_{\alpha,k}}\cos(\omega h) - \frac{q_i}{\sigma_{\alpha,k}}\right) \ge 1, \text{ for all } \alpha, n, \omega, h \text{ and } k \text{ we have } \xi_1 \le \xi_0. \tag{12}$$

and

$$\xi_n \le \xi_{n-1} + \sum_{j=2}^{n}\omega_j^{(\alpha)}\left(\xi_{n-j} - \xi_{n-j+1}\right), n \ge 2. \tag{13}$$

Thus, for *n=2*, the last inequality implies

$$\xi_2 \le \xi_1 + \omega_2^{(\alpha)}\left(\xi_0 - \xi_1\right)$$

Again repeating the above process, we can get

$\xi_j \le \xi_{j-1}$, *j=1,2,...n-1.*

From Eq.(13), we finally have

$$\xi_n \le \xi_{n-1} + \sum_{j=2}^{n}\omega_j^{(\alpha)}\left(\xi_{n-j} - \xi_{n-j+1}\right) \le \xi_{n-j}.$$

Since each term in the summation is negative, it shows that the inequalities Eq.(12) and Eq.(13) imply

$$\xi_n \le \xi_{n-1} \le \xi_{n-2} \le \ldots \le \xi_1 \le \xi_0.$$

Thus,

$\xi_n = \left|U_j^n\right| \le \xi_0 = \left|U_j^0\right| = \left|f_j\right|$, which entails $\left\|U_j^n\right\| \le \left\|f_j\right\|$, and we have stability.

5.Half-Sweep Preconditioned SOR Method

In relation to the linear system in Equation (8), it is clear that the characteristics of its coefficient matrix are large scale and sparse. Basically in first Section, many scientist have discussed various iteration methods such as (Gunawardena et al., 1991; Hackbusch, 2016; Saad, 2003; Salkuyeh & Shamsi, 2012; Young, 1971). To get numerical solving of the tridiagonal linear system (8), we proposed the Half-Sweep Preconditioned Successive Over Relaxation (HSPSOR) iteration method (Shen et al., 2009; Sunarto, Sulaiman, & Saudi, 2015), to solve any linear systems.

Before applying the HSPSOR iteration method, we need to transform the original linear system (9) into the preconditioned linear system

$$A^* \underset{\sim}{Z} = \underset{\sim}{f}^* \tag{14}$$

where, $A^* = PAP^T$, $\underset{\sim}{f}^* = P\underset{\sim}{f}$, $\underset{\sim}{Z} = P^T \underset{\sim}{y}$.

Actually, the matrix P is called a preconditioned matrix and defined as (Gunawardena et al., 1991; Sunarto et al., 2015)

$$P = I + S \tag{15}$$

where

$$S = \begin{bmatrix} 0 & -r_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -r_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -r_3 & 0 & 0 \\ 0 & 0 & O & O & O & 0 \\ 0 & 0 & 0 & 0 & 0 & -r_{m-1} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}_{(m-1)x(m-1)}$$

and the matrix I is an identical matrix. To develop Half-Sweep PSOR iteration method, let consider the coefficient matrix A^* in equation (9) can be defined as summation of the matrices

$$A^* = Di - Lo - Va \tag{16}$$

with *Di*, *Lo* and *Va* are diagonal, lower triangular and upper triangular matrices. By using Eq.(14) and (16) , the formulation of Half-Sweep PSOR iteration method can be rewritten as (Shen et al., 2009; Sunarto et al., 2015)

$$\underset{\sim}{y}^{(k+1)} = (D - \omega L)^{-1}[(1-\omega)D + V\omega]\underset{\sim}{y}^{(k)} + (D - \omega L)^{-1}\underset{\sim}{f}^* \tag{17}$$

where $\underset{\sim}{y}^{(k+1)}$ represents an unknown vector at *(k+1)*th iteration. The application of the HSPSOR iteration method can be explained in this below.

HSPSOR method

i. First $\tilde{U} \leftarrow 0$ and $\varepsilon \leftarrow 10^{-10}$.
ii. Then $j = 1,2,\ldots,n$ Implement
   a. with $i = 1,2,\ldots,m-1$ calculate

   $$\underset{\sim}{y}^{(k+1)} = (D - \omega L)^{-1}[(1-\omega)D + V\omega]\underset{\sim}{y}^{(k)} + (D - \omega L)^{-1}\underset{\sim}{f}^*$$

   $$\underset{\sim}{Z}^{(k+1)} = P^T \underset{\sim}{y}^{(k+1)}$$

   b. Next to the convergence test. If the convergence criterion i.e $\left\| \underset{\sim}{y}^{(k+1)} - \underset{\sim}{y}^{(k)} \right\| \le \varepsilon = 10^{-10}$ is satisfied, go to Step (iii). If not go back to Step (a).
iii Display approximate equation solutions.

6.Evolution Of Numerical Problems

By using approximation Eq.(7), we consider one problem of the TFPDE's to test the performance of the Fulll-Sweep Preconditioned Successive Over-Relaxation (FSPSOR) and Half-Sweep Preconditioned Successive Over-Relaxation (HSPSOR) iteration methods. In order to compare the performance of these proposed iteration methods, three criteria have been considered such as K (number of iterations), Time (in seconds) and Max Absolute Error at three values, where value of *α = 0.25, α = 0.50 and α = 0.75*. For application of three iterative schemes, the convergence test considered the tolerance error as $\varepsilon = 10^{-10}$.

Let us examine the TFPDE's initial boundary conditions value problem (Ali, S E, Ozgur & Korkmaz, 2013)

$$\frac{\partial^\alpha Z(y,r)}{\partial r^\alpha} = \frac{\partial^2 Z(y,r)}{\partial y^2}, \quad 0 < \alpha \le 1, 0 \le y \le \gamma, r > 0 , \tag{18}$$

with the boundary value conditions are $Z(0,r) = \frac{2kr^\alpha}{\Gamma(\alpha+1)}$, $Z(l,r) = l^2 + \frac{2kr^\alpha}{\Gamma(\alpha+1)}$, (19) and the initial value condition $Z(y,0) = y^2$. (20)

Overall results of evolution of numerical problem for equation (18), obtained from application of FSPSOR and HSPSOR iteration methods are recorded in Table 1, where value of mesh sizes, *m* = 128, 256, 512, 1024, and 2048.

7. Conclusions

To In order to get the numerical solution of the TFPDE's problems equation, this study give the derivation of the implicit Caputo's finite difference approximation equations in which this approximation equation leads a tridiagonal linear system. Via all experimental results by imposing the FSPSOR and HSPSOR iteration methods, it is clear at $\alpha = 0.25$ that K (number of iterations) have declined approximately by 64.27-96.14% conforms to the HSPSOR iteration method compared with FSPSOR methods. Then for Time, application of HSPSOR iteration method are much faster about 25.84-94.18% than the FSPSOR methods. It can be also observed in Table 1 that the HSPSOR method requires the least amount for K (number of iterations) and Time at $\alpha = 0.25$ as compared with FSPSOR iteration methods. According to the accuracy of both iteration methods, it can be concluded that their numerical solutions of TFPDE's are in good agreement.

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TABLE 1. Comparison between number of iterations (K), the computational time (seconds) and maximum absolute errors (MAE) for both iterative methods at $\alpha = 0.25, 0.50, 0.75$

M	Method	α = 0.25			α = 0.50			α = 0.75		
		K	Time	MAE	K	Time	MAE	K	Time	MAE
128	**FSPSOR**	281	2.24	9.95e-05	229	1.95	9.84e05	164	1.63	1.29e-04
	HSPSOR	156	1.21	9.95e-05	137	1.06	9.84e05	132	0.99	1.30e-04
256	**FSPSOR**	1428	16.9	9.95e-05	1171	12.61	9.84e-05	814	8.90	1.29e-04
	HSPSOR	778	4.89	9.95e-05	486	3.31	9.84e-05	211	1.90	1.30e-04
512	**FSPSOR**	5524	113.86	9.96e-05	4520	91.37	9.84e-05	3137	61.98	1.30e-04
	HSPSOR	3006	32.07	9.96e-05	1929	20.66	9.84e-05	931	10.60	1.30e-04
1024	**FSPSOR**	20574	817.59	9.98e-05	16842	662.23	9.87e-05	11695	456.23	1.30e-04
	HSPSOR	11216	236.40	9.98e-05	7245	155.08	9.87e-05	3551	74.13	1.30e-04
2048	**FSPSOR**	75580	3043.59	1.01e-04	61941	2894.70	9.90e-05	43070	977.10	1.30e-04
	HSPSOR	40270	1835.90	9.98e-05	25487	1320.76	9.84e-05	13208	552.95	1.30e-04

Biography / Biographies

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