A Newton-Modified Weighted Arithmetic Mean Solution of Nonlinear Porous Medium Type Equations

by Jackel Chew

Article

A Newton Modified Weighted Arithmetic Mean Solution of Nonlinear Porous Medium Type Equations

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Abstract: The mathematical theory behind the porous medium type equation is well-developed and produces many applications to the real world. The research and development of the fractional nonlinear porous medium models also progressed significantly in recent years. An efficient numerical method to solve porous medium models needs to be investigated so that the symmetry of the designed method can be extended to future fractional porous medium models. This paper contributes a new numerical method called Newton Modified Weighted Arithmetic Mean (Newton-MOWAM). The solution of the porous medium type equation is approximated by using a finite difference method. Then, the Newton method is applied as a linearization approach to solving the system of nonlinear equations. Since the system to be solved is large, high computational complexity is regulated by the MOWAM iterative method. Newton-MOWAM is formulated technically based on the matrix structure of the system. Some initial-boundary value problems with a different type of nonlinear diffusion term are presented. As a result, the Newton-MOWAM showed a significant improvement in the computation efficiency compared to the developed standard Weighted Arithmetic Mean iterative method. The analysis of efficiency, measured by the reduced number of iterations and computation time, is reported along with the convergence analysis.

Keywords: porous medium type equation; nonlinear diffusion; Newton method; finite difference method; iterative method; weighted arithmetic mean

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1. Introduction

Porous medium type equation is one of the classes of nonlinear parabolic evolution equation. This class of partial differential equations appears in the description and modelling of natural physical phenomena such as gas diffusion, fluid flow and heat propagation. The mathematical theory behind the nonlinear porous medium type equation is well-developed and produces many applications to the real world [1]. A porous medium type equation is used to investigate the diffusion of reactant gases across a thin layer of porous material. For instance, [2] developed a mathematical model to describe the flow of a mixture of ideal gases in a highly porous electrode for fuel cell engineering. In their developed model, a porous medium type equation is used to simulate the evolution of the gas mixture.

Porous medium type equation is also one of the important mathematical models used in underground oil production. The permeability of rock above the underground oil wells, which represents the porous medium for the oil diffusion, determines the maximum capacity of oil to be produced over a fixed period. Moreover, underground oil production is affected by a phenomenon called instability during the oil extracting process. The instability phenomenon appears in the form of irregular trembling fingers caused by the immiscibility of water and oil. One article has proposed a mathematical model using the formulated porous medium type equation to simulate the instability phenomenon [3]. In addition to that, porous medium type equation has its importance in life sciences such as bacteria biofilms growth model [4], cell population dynamics model [5], and cell to cell adhesion model [6].

Real world phenomena modelled by the porous medium type equation contains arbitrary functions or constants. These functions and constants, which act as the model parameters, may be specified through several experiments or applying the well-established physical laws. However, some forms of these parameters are assumed in most numerical studies. Some numerical experiments also use random values for the parameters to investigate the accuracy and efficiency of the proposed numerical approach. Many examples of porous medium type equations with arbitrary parameters can be obtained in [7]. Besides that, the research and development of the fractional nonlinear porous medium type equation have progressed significantly in recent years and is recommended to read [8].

Many different numerical methodologies have been presented and applied to solve the mathematical models of porous medium types. The models are varied in the degree of non-linearity, the number of derivatives terms and the type of diffusion term. Thus, an efficient numerical method to solve porous medium models needs to be investigated so that the symmetry of the designed method can be extended to future fractional porous medium models. A new numerical method called Newton Modified Weighted Arithmetic Mean (Newton-MOWAM) is introduced and studied in this paper. The solution of the porous medium type equation is approximated by using a finite difference method. Then, the Newton method is applied as a linearization approach to solving the system of nonlinear equations. The nonlinear equations generated by the finite difference approximation to a nonlinear partial differential equation usually possess a high computational complexity. In order to overcome the high computational complexity to solve the nonlinear problems, ~~Hence, this paper presents~~ the complexity regulation by the Newton-MOWAM ~~which~~ is formulated technically based on the matrix structure of the system. Hence, the contribution of this paper is to propose the Newton-MOWAM method to solve the nonlinear porous medium type equations efficiently with rigorously analysis of convergence by developing a theorem and its corresponding proof.

2. Preliminary

The basic idea behind the numerical method to resolve the porous medium type equation can be represented in the form of a linear system as follows

$$\mathcal{M}\widehat{\varphi} = \widehat{F}, \tag{1}$$

where $\mathcal{M}$ is a coefficient matrix, $\widehat{F}$ is a given vector, and $\widehat{\varphi}$ signifies the unknown vector to be calculated. Basically, methods for solving Equation (1) can be categorized into two main streams i.e. direct and iterative methods. In principle, the direct method is the solution of Equation (1) is determined through a finite number of arithmetic operations which without consideration of round-off errors. In contrast to direct method, iterative method generates a chronological order of approximations to the solution by recurrence application of the same computational prototype at each iteration. When the size of the linear system is large, iterative method is always preferable to be used compared to direct method. Direct methods have the complexity of order $O(m^3)$, where m is the ~~mesh size~~ order of a matrix. In contrast, for iterative methods, the work per one iteration is essentially the matrix-vector multiplication, which for sparse matrices is sometimes much less than $O(m^2)$. In addition, the number of iterations needed for the convergence, when using a suitable preconditioner, is usually much less than $O(m)$. Thus, the overall work is much less than $O(m^3)$, which means it is fast convergence. According to the book by [9], sometimes we do not need to solve the system of equations exactly. For example, a few methods for solving nonlinear system of equations, in which each iteration involves a linear system to solve. Frequently, it is sufficient to solve the linear system within the nonlinear iteration only to a low degree of accuracy. Direct methods cannot accomplish this because, by definition, to obtain a solution, the process must be completed. There is no notion of early termination or an inexact solution.

A general technique to devise iterative method is based on an additive splitting of matrix $\mathcal{M}$ into $\mathcal{M} = \mathcal{P} - \mathcal{Q}$ where $\mathcal{M}$ must be a nonsingular matrix. Then, the corresponding iterative scheme for solving linear system as in Equation (1) is of the form

$$\widehat{\varphi}^{(\ell+1)} = \mathcal{P}^{-1}\mathcal{Q}\widehat{\varphi}^{(\ell)} + \mathcal{P}^{-1}\widehat{F}, \quad \ell = 0, 1, 2, ..., \tag{2}$$

with the new iterate $\widehat{\varphi}^{(\ell+1)}$, previous iterate $\widehat{\varphi}^{(\ell)}$, vector $\mathcal{P}^{-1}\widehat{F}$ and iteration matrix $\mathcal{P}^{-1}\mathcal{Q}$. Based on the iterative form shown by Equation (2), there are two main types of iterative methods, namely stationary iterative and nonstationary iterative methods. Both types of methods are classified based on the nature of iteration matrix $\mathcal{P}^{-1}\mathcal{Q}$ and vector $\mathcal{P}^{-1}\widehat{F}$. For stationary methods, the iteration matrix $\mathcal{P}^{-1}\mathcal{Q}$ and vector $\mathcal{P}^{-1}\widehat{F}$ remain constant throughout the iteration process, while a new iteration matrix $\mathcal{P}^{-1}\mathcal{Q}$ and vector $\mathcal{P}^{-1}\widehat{F}$ are generated in every step of the nonstationary iterative methods.

In this paper, several stationary iterative methods are reviewed i.e. Alternating Group Explicit [10], Modified Alternating Group Explicit [11], Iterative Alternating Decomposition Explicit [12], Block SOR [13] and Weighted Mean (WM) [14,15]. Among these names of stationary iterative method, the concept from the WM method is adopted to develop a new efficient numerical method for porous medium type equation. The WM method is a group of algorithms that have been widely used to solve matrix problems efficiently. One of the WM methods to be investigated in this paper is, the Arithmetic Mean (AM) iterative method [16]. In the previous literature, the AM method and its variants have been studied and scrutinized on linear and nonlinear systems arising from a variety of scientific problems such as [17–22]. In the study by [14] and [16], the numerical simulations showed that AM method compare favourably against the existing nonstationary iterative method, particularly when the iteration matrix is strongly asymmetric. Further, Aruchunan et al. [23] introduced a modified AM (MAM) method by initiating second weighted optimal value to improve the convergence of AM algorithm via solving fourth order integro-differential equations. However, the author did not discuss the convergence theorem and its corresponding proofs in the MAM method. Therefore, in this paper, the MAM method is extended together with Newton method for solving nonlinear porous medium type equation accompanied by the complete convergence theorem and proof. The complete procedures on developing Newton-MOWAM is described in the following segments.

3. Materials and Methods

3.1. Porous medium type equation

This paper considers the numerical solution of a general one-dimensional porous medium type equation as follows [7],

$$\frac{\partial w}{\partial t} = \frac{\partial}{\partial x}\left[f(w)\frac{\partial w}{\partial x}\right], \tag{3}$$

where the function $f(w)$ represents the type of diffusion term which is depending on the nonlinear physical phenomenon to be modelled. For example, when $f(w) = w^{\alpha}$ with the real number α, the simplest form of porous medium equation used to model the instability phenomenon in oil recovery can be obtained [3]. Using a finite difference method, the solution domain of Equation (3) can be restricted about a unit square containing numerous number of well distributed mesh unknown grid points. The restriction is usually assumed by imposing suitable initial condition $w(x,0) = w_0(x), 0 \leq x \leq X$ and the Dirichlet boundary condition $w(0,t) = a(t), w(X,t) = b(t), 0 \leq t \leq T$.

3.2. Finite difference approximation to porous medium type equation

To formulate the finite difference approximation equation to Equation (3), it is more appropriate to differentiate the right hand side of Equation (3) into

$$\frac{\partial w}{\partial t} = f'(w)\left(\frac{\partial w}{\partial x}\right)^2 + f(w)\frac{\partial^2 w}{\partial x^2}, \tag{4}$$

where $f'(w)$ is the derivative of the smooth function $f(w)$.

To set up the network of mesh grid points as the approximate solutions of Equation (4), we define the approximate solution by $\mathcal{W}(x,t) = \mathcal{W}(i\Delta x, j\Delta t), i = 0, ..., m, j = 0, ..., n$, where Δx and Δt are the spatial and temporal mesh step sizes respectively. In this work, a square-shaped domain is considered with the finite interval in space, $[0, X]$ is divided into $m-1$ intervals that yields m mesh unknown grid points and the finite interval in time, $[0, T]$ is divided into $n-1$ intervals. By substituting a backward time operator and a second order center space operator into the time and space derivatives in Equation (4) respectively, the implicit finite difference approximation to Equation (3) can be formulated as

$$\begin{aligned}\frac{\mathcal{W}_{i,j+1} - \mathcal{W}_{i,j}}{\Delta t} &= f'(\mathcal{W}_{i,j+1})\left(\frac{\mathcal{W}_{i+1,j+1} - \mathcal{W}_{i-1,j+1}}{2\Delta x}\right)^2 \\ &+ f(\mathcal{W}_{i,j+1})\left(\frac{\mathcal{W}_{i+1,j+1} - 2\mathcal{W}_{i,j+1} + \mathcal{W}_{i-1,j+1}}{\Delta x^2}\right),\end{aligned} \tag{5}$$

and simplified into

$$\begin{aligned}&\mathcal{W}_{i,j+1} - \frac{\Delta t}{4\Delta x^2} f'(\mathcal{W}_{i,j+1})(\mathcal{W}_{i+1,j+1} - \mathcal{W}_{i-1,j+1})^2 \\ &- \frac{\Delta t}{\Delta x^2} f(\mathcal{W}_{i,j+1})(\mathcal{W}_{i+1,j+1} - 2\mathcal{W}_{i,j+1} + \mathcal{W}_{i-1,j+1}) = \mathcal{W}_{i,j}.\end{aligned} \tag{6}$$

Since the discrete finite difference approximation equation shown by Equation (6) is nonlinear, it yields a system of nonlinear equations when a fixed number of mesh unknown points is considered. The approximate solution of the porous medium type equation needs to be obtained through the solution of the system of nonlinear equations. Instead of solving the high computational complexity system directly, Newton method is adopted as an effective linearization approach to solve the system of nonlinear equations.

3.3. Newton method

To apply the Newton method, Equation (6) needs to be rearranged into a function as follows

$$\begin{aligned}F_{i,j+1} = &\frac{\Delta t}{\Delta x^2} f(\mathcal{W}_{i,j+1})\mathcal{W}_{i+1,j+1} - \left(1 + \frac{2\Delta t}{\Delta x^2} f(\mathcal{W}_{i,j+1})\right)\mathcal{W}_{i,j+1} \\ &+ \frac{\Delta t}{\Delta x^2} f(\mathcal{W}_{i,j+1})\mathcal{W}_{i-1,j+1} + \frac{\Delta t}{4\Delta x^2} f'(\mathcal{W}_{i,j+1})(\mathcal{W}_{i+1,j+1} - \mathcal{W}_{i-1,j+1})^2 + \mathcal{W}_{i,j}.\end{aligned} \tag{7}$$

When a Jacobian matrix with respect to three unknown mesh points, i.e. $\mathcal{W}_{i-1,j+1}, \mathcal{W}_{i,j+1}$ and $\mathcal{W}_{i+1,j+1}$, is derived, a tridiagonal coefficient matrix can be formed. Hence, the associated system of linear equations developed from the discretized Equation (3) is written in the form of

$$\mathcal{M}\widehat{\mathcal{U}} = \widehat{\mathcal{F}}, \tag{8}$$

where

$$\mathcal{M} = \begin{bmatrix} \mathcal{D}_1 & \mathcal{V}_1 & & & & \\ \mathcal{L}_2 & \mathcal{D}_2 & \mathcal{V}_2 & & & \\ & \vdots & \vdots & \vdots & & \\ & & \vdots & \vdots & \vdots & \\ & & & \mathcal{L}_{m-2} & \mathcal{D}_{m-2} & \mathcal{V}_{m-2} \\ & & & & \mathcal{L}_{m-1} & \mathcal{D}_{m-1} \end{bmatrix}, \tag{9}$$

with the following tridiagonal coefficients:

$$\begin{aligned}
\mathcal{L}_i &= \frac{\Delta t}{\Delta x^2} f(\mathcal{W}_{i,j+1}) - \frac{\Delta t}{2\Delta x^2} f'(\mathcal{W}_{i,j+1})(\mathcal{W}_{i+1,j+1} - \mathcal{W}_{i-1,j+1}), \\
&\quad i = 2, \ldots, m-2, m-1, \\
\mathcal{D}_i &= \frac{\Delta t}{\Delta x^2} f'(\mathcal{W}_{i,j+1})\mathcal{W}_{i+1,j+1} - \left((1 + \frac{2\Delta t}{\Delta x^2} f(\mathcal{W}_{i,j+1})) + (\frac{2\Delta t}{\Delta x^2} f'(\mathcal{W}_{i,j+1}))(\mathcal{W}_{i,j+1}) \right) \\
&\quad + \frac{\Delta t}{\Delta x^2} f'(\mathcal{W}_{i,j+1})\mathcal{W}_{i-1,j+1} + \frac{\Delta t}{4\Delta x^2} f''(\mathcal{W}_{i,j+1})(\mathcal{W}_{i+1,j+1} - \mathcal{W}_{i-1,j+1})^2, \\
&\quad i = 1, 2, \ldots, m-2, m-1, \\
\mathcal{V}_i &= \frac{\Delta t}{\Delta x^2} f(\mathcal{W}_{i,j+1}) + \frac{\Delta t}{2\Delta x^2} f'(\mathcal{W}_{i,j+1})(\mathcal{W}_{i+1,j+1} - \mathcal{W}_{i-1,j+1}), \\
&\quad i = 1, 2, \ldots, m-2,
\end{aligned}$$

We define the corrector to the approximate solutions of (3) as

$$\widehat{U} = \mathcal{W}^{(\ell+1)}_{i,j+1} - \mathcal{W}^{(\ell)}_{i,j+1}, \quad i = 1, 2, \ldots, m-2, m-1, j = 0, \ldots, n-1, \tag{10}$$

with the iteration index, ℓ and $\widehat{F} = (F_{1,j+1}, F_{2,j+1}, \ldots, F_{m-2,j+1}, F_{m-1,j+1})^T$.

3.4. Newton Modified Weighted Arithmetic Mean

Aforemention in the Section 1, the major contribution of this research is to establish new numerical method called Newton-MOWAM with its complete convergence theorem and proof. The Newton-MOWAM is developed by proposing two optimal parameters on the existing Weighted Arithmetic Mean (WAM) iterative method. Following to that, the developed Newton-MOWAM method will be investigated by solving the system of linear equation in matrix form as in Equation (8). ~~Therefore, in this section,~~ Therefore, in the next few subsections, the formulations of novel Newton-MOWAM method will be explained by demonstrating its convergence theorems and proofs. The iteration procedure for Newton-MOWAM method consists of two loops, where in Loop 1, an independent system, $\hat{\varphi}^F$ is involved meanwhile in Loop 2 independent system of $\hat{\varphi}^B$ is involved. This work assume the number of ~~mesh~~ unknown points m is an even number.

3.5. Weighted optimal value identification

The weighted optimal value is crucial to accelerate the convergence process for the iterative methods [24]. Therefore, the Newton-MOWAM method has weighted optimal values for rapid convergence. In this research, another new weighted optimal value, Ω_2 is initiated for Loop 2. The optimal value Ω_2 was set based on trial with least number of iterations after the best Ω_1 is fixed. The execution of these two weighted optimal values improved the convergence rate of the Newton-MOWAM compared to the standard WAM methods. On condition that $\Omega_1 = \Omega_2$, the Newton-MOWAM method is equivalent to the conventional WAM methods. The weighted optimal values Ω_1 and Ω_2 have values between 0 and 2 in the range.

3.6. Matrix Splitting

According to [25], the unique splittings can be used to represent a broad class of iterative methods hence, splittings can be used to represent both outer and inner iterations with its condition of generality. Therefore, in this paper, the generated monotone coefficient matrix splittings for both stationary and nonstationary methods, which can be converged for inner iterations were also demonstrated. In order to conduct a methodical analysis of convergence conditions, matrix splitting is required. Hence, let ~~consider~~ the matrix splitting of $\mathcal{M}$ be expressed as

$$\mathcal{M} = \mathcal{H}_r - \mathcal{K}_r, \quad r = 1,2, \tag{11}$$

where $\mathcal{H}_1$ and $\mathcal{H}_2$ are given by

$$\mathcal{H}_1 = \begin{bmatrix} \sigma_{1,1} & \sigma_{1,2} & & & & & \\ \sigma_{2,1} & \sigma_{2,2} & & & & & \\ & & \sigma_{3,3} & \sigma_{3,4} & & & \\ & & \sigma_{4,3} & \sigma_{4,4} & & & \\ & & & & \ddots & & \\ & & & & & \sigma_{m-2,m-2} & \sigma_{m-2,m-1} \\ & & & & & \sigma_{m-1,m-2} & \sigma_{m-1,m-1} \end{bmatrix},$$

and

$$\mathcal{H}_2 = \begin{bmatrix} \sigma_{1,1} & & & & & & \\ & \sigma_{2,2} & \sigma_{2,3} & & & & \\ & \sigma_{3,2} & \sigma_{3,3} & & & & \\ & & & \ddots & & & \\ & & & & \sigma_{m-3,m-3} & \sigma_{m-3,m-2} & \\ & & & & \sigma_{m-2,m-3} & \sigma_{m-2,m-2} & \\ & & & & & & \sigma_{m-1,m-1} \end{bmatrix},$$

respectively. Consequently, $\mathcal{K}_r$ is expressed as

$$\mathcal{K}_r = \mathcal{H}_r - \mathcal{M}, \quad r = 1,2. \tag{12}$$

The MOWAM iterative method is can be defined as follows

$$\left.\begin{aligned} (\mathcal{D} - \Omega_1\mathcal{L})\widehat{\varphi}^{(F)} &= ((1-\Omega_1)\mathcal{D} + \Omega_1\mathcal{V})\widehat{\varphi}^{(\ell)} + \Omega_1\widehat{f} \\ (\mathcal{D} - \Omega_2\mathcal{V})\widehat{\varphi}^{(B)} &= ((1-\Omega_2)\mathcal{D} + \Omega_2\mathcal{L})\widehat{\varphi}^{(\ell)} + \Omega_2\widehat{f} \\ \widehat{\varphi}^{(\ell+1)} &= \frac{1}{2}(\widehat{\varphi}^{(F)} + \widehat{\varphi}^{(B)}) \end{aligned}\right\} \quad \ell = 0,1,\ldots, \tag{13}$$

where $(\mathcal{D} - \Omega_1\mathcal{L})$, $(\mathcal{D} - \Omega_2\mathcal{V})$, $((1-\Omega_1)\mathcal{D} - \Omega_1\mathcal{L})$ and $((1-\Omega_2)\mathcal{D} - \Omega_2\mathcal{V})$ are nonsingular triangular matrices, Ω_1 and Ω_2 are weighted optimal values, $\widehat{\varphi}^{(\ell)}$ is an initial vector at the ℓth iteration, meanwhile, $\widehat{\varphi}^{(F)}$ and $\widehat{\varphi}^{(B)}$ are independent solution of forward and backward iterations, respectively.

From Equation (13), the Newton-MOWAM iterative scheme for the linear system of Equation (8) is

$$\widehat{\varphi}^{(\ell+1)} = \mathcal{Z}_{MOWAM}\widehat{\varphi}^{(\ell)} + z_{MOWAM}\widehat{F}, \quad \ell = 0,1,2,\ldots, \tag{14}$$

where

$$\mathcal{Z}_{MOWAM} = \frac{1}{2}\Big[(\mathcal{D} - \Omega_1\mathcal{L})^{-1}((1-\Omega_1)\mathcal{D} + \Omega_1\mathcal{V}) + (\mathcal{D} - \Omega_2\mathcal{V})^{-1}((1-\Omega_2)\mathcal{D} + \Omega_2\mathcal{L})\Big]$$

is a square matrix. Meanwhile,

$$z_{MOWAM} = \frac{1}{2}\Big[\Omega_1(\mathcal{D}-\Omega_1\mathcal{L})^{-1} + \Omega_2(\mathcal{D}-\Omega_2\mathcal{V})\Big]$$

is a coefficient of load vector $\widehat{F}$.

For ~~guarantees~~ guaranteed convergence of the MOWAM iteration, the general condition for solving the Equation (8) is proven in the following Theorem.

Theorem 1. *Given an $(m-1)\times(m-1)$ nonsingular tridiagonal matrix with diagonally dominant $\mathcal{M}$ with its components $\sigma_{i,i} > 0$, for $i = 1, 2, \ldots, m-2, m-1$, and*

$$\mathcal{M} = \mathcal{H}_1 - \mathcal{K}_1 = \mathcal{H}_2 - \mathcal{K}_2$$

where matrices $(\mathcal{H}_1)^{-1}$ and $(\mathcal{H}_2)^{-1}$ are nonsingular with $\|(\mathcal{H}_1)^{-1}\| \geq 0$, $\|\mathcal{K}_1\| \geq 0$ and $\|(\mathcal{H}_2)^{-1}\| \geq 0$, $\|\mathcal{K}_2\| \geq 0$. The Newton-MOWAM iteration shown by Equation (14) is convergent for $0 < \Omega_1 < 2$ and $0 < \Omega_2 < 2$.

Proof. By hypothesis, $\mathcal{M}$ is an $(m-1)\times(m-1)$ regular matrix. Since $\mathcal{H}_1 = \mathcal{D} - \Omega_1\mathcal{L}$ and $\mathcal{H}_2 = \mathcal{D} - \Omega_2\mathcal{V}$ are positive definite or strictly diagonally dominant matrices for $0 < \Omega_1 < 2$ and $0 < \Omega_2 < 2$, the matrices $\mathcal{K}_1 = (1-\Omega_1)\mathcal{D} + \Omega_1\mathcal{V}$ and $\mathcal{K}_2 = (1-\Omega_2)\mathcal{D} + \Omega_1\mathcal{L}$ are triangular and nonnegative. As

$$\mathcal{H}_1 - \mathcal{K}_1 = \mathcal{H}_2 - \mathcal{K}_2 = \mathcal{M},$$

then we have

$$\mathcal{Q} = \frac{1}{2}(\mathcal{H}_1)^{-1}\mathcal{K}_1 + \frac{1}{2}(\mathcal{H}_2)^{-1}\mathcal{K}_2 = \mathcal{I} - \left[\frac{1}{2}(\mathcal{H}_1)^{-1} + \frac{1}{2}(\mathcal{H}_2)^{-1}\right]\mathcal{M}, \tag{15}$$

or also can be written as

$$\frac{1}{2}(\mathcal{H}_1)^{-1} + \frac{1}{2}(\mathcal{H}_2)^{-1} = (\mathcal{I} - \mathcal{Q})(\mathcal{M})^{-1}. \tag{16}$$

The proof of the theorem runs parallel to a standard proof given in [26]. Since $\mathcal{Q} = (\mathcal{H}_r)^{-1}\mathcal{K}_r$, then the spectral radius is

$$\rho_{MOWAM}(\mathcal{Q}) < 1. \tag{17}$$

Therefore, the conditions of $0 < \Omega_1 < 2$ and $0 < \Omega_2 < 2$ for the Newton-MOWAM iteration shown by Equation (14) is converged for any initial vector $\widehat{\varphi}^{(0)}$. Hence, the Theorem 1 is proved. □

4. Numerical Results

4.1. Porous medium type equation selected problems

In order to evaluate the efficacy of the Newton-MOWAM iterative method, two examples of one-dimensional porous medium type initial-boundary value problems are selected with different levels of difficulties. Below are the following problems that we consider in the numerical experiment.

Problem 1:

$$\frac{\partial w}{\partial t} = \frac{\partial}{\partial x}\left[(Aw^2 + Bw)\frac{\partial w}{\partial x}\right], \tag{18}$$

where A and B are arbitrary constants. The exact solution used for the accuracy checking is given by [7]

$$w(x,t) = \pm\frac{x+C}{\sqrt{D-4At}} - \frac{B}{2A}, \tag{19}$$

with arbitrary constants C and D. We chose $A = -0.5, B = 1, C = 1,$and $D = 5$ as the specification of Problem 1.

Problem 2:

$$\frac{\partial w}{\partial t} = \frac{\partial}{\partial x}\left[\frac{A}{w^2 + B^2}\frac{\partial w}{\partial x}\right], \tag{20}$$

where A and B are arbitrary constants. Exact solution is given by [7]:

$$w(x,t) = \frac{Bx}{\sqrt{C - \frac{2At}{B^2} - x^2}}, \tag{21}$$

with an arbitrary constant C and the values that we have selected are $A = 1, B = 5,$and $C = 4$.

In order to test the applicability of the Newton-MOWAM method to solve a higher degree of difficulty problem, a two-dimensional porous medium type initial-boundary value problem is proposed. Noted that the formulation of a finite difference approximation to a two-dimensional porous medium type equation can be made by extending the formulation in Subsection 3.2 to another spatial direction. The full formulation of a finite difference approximation to a two-dimensional porous medium type equation can be referred in one of the authors' previous work [27]. Below is the following two-dimensional porous medium type initial-boundary value problem.

Problem 3:

$$\frac{\partial w}{\partial t} = \frac{\partial}{\partial x}\left[w^5\frac{\partial w}{\partial x}\right] + \frac{\partial}{\partial y}\left[w^5\frac{\partial w}{\partial y}\right]. \tag{22}$$

Exact solution used to test the accuracy of the solution is based on [28]:

$$w(x,y,t) = \sqrt[4]{0.8x + 0.8y + 1.6t}. \tag{23}$$

In executed numerical experiment, the efficiency of the Newton-MOWAM iterative method is compared to Newton-WAM iterative method, which was developed independently by using the combination of Newton method and the existing WAM iterative method. There are two criteria used to measure the difference in terms of efficiency between Newton-MOWAM and Newton-WAM methods: the number of iterations (ℓ_{max}) and the program execution time recorded in seconds (s). Besides that, the accuracy of the two tested numerical methods is taken into account by recording the maximum values of the absolute errors.

4.2. C program implementation and algorithm

The prototype of proposed Newton-MOWAM and the standard Newton-WAM methods are designed and developed by using C programming. The C programming language is used because of its good coding organization and comprehension in the field of numerical analysis. Furthermore, with the basic C programming concept, a correct calculation of the number of iterations and the execution time can be taken. For the comparison analysis, the implementation of the Newton-MOWAM and Newton-WAM on Problem ~~1 and 2~~ 1, 2 and 3 is made independently. The number of iterations, the execution time, and the maximum absolute error are recorded using five different ~~mesh sizes m~~ orders of a matrix, that is $m = 64$, 128, 256, 512, and 1024 for Problem 1 and 2, while $m \times m = 16 \times 16$, 32×32, 64×64, 128×128, and 256×256 for Problem 3. The results from these different ~~mesh sizes~~ orders of a matrix are used for consistency verification. Furthermore, the numerical solution can only be obtained through the running C program after the iteration process was completed successfully. Since the C program code is copyrighted, below is the Newton-MOWAM algorithm used for the program design and development.

Algorithm 1: Newton-MOWAM iterative method:

1. Define Ω_1 and Ω_2;
2. Define $w_0(x)$ for $0 \leq x \leq X$;
3. Define $a(t)$ and $b(t)$, for $0 \leq t \leq T$;
4. Construct $\mathcal{M}\widehat{U} = \widehat{F}$;
5. For $j = 0, i = 1, 2, \ldots, m-2, m-1$, set the initial guess $\mathcal{W}^{(0)}_{i,j+1} = 1$;
6. Initialize $\widehat{U}^{(0)} = 0$;
7. For $\ell = 0, 1, 2, \ldots$;
   $\widehat{\varphi}^{(\ell)} = \widehat{U}^{(\ell)}$;
   $(\mathcal{D} - \Omega_1\mathcal{L})\widehat{\varphi}^{(F)} = ((1-\Omega_1)\mathcal{D} + \Omega_1\mathcal{V})\widehat{\varphi}^{(\ell)} + \Omega_1\widehat{F}$;
   $(\mathcal{D} - \Omega_2\mathcal{V})\widehat{\varphi}^{(B)} = ((1-\Omega_2)\mathcal{D} + \Omega_2\mathcal{L})\widehat{\varphi}^{(\ell)} + \Omega_2\widehat{F}$;
   $\widehat{\varphi}^{(\ell+1)} = \frac{1}{2}(\widehat{\varphi}^{(F)} + \widehat{\varphi}^{(B)})$;
   $\widehat{U}^{(\ell+1)} = \widehat{\varphi}^{(\ell+1)}$;
8. Convergence criterion $\left|\widehat{U}^{(\ell+1)} - \widehat{U}^{(\ell)}\right| \leq 10^{-10}$;
9. Compute $\mathcal{W}^{(\ell+1)} = \mathcal{W}^{(\ell)} + \widehat{U}^{(\ell+1)}$;
10. Convergence criterion $\left|\widehat{F}(\mathcal{W}^{(\ell+1)}) - \widehat{F}(\mathcal{W}^{(\ell)})\right| \leq 10^{-10}$;
11. Go to the next time step $j++$;
12. Display the numerical outputs.

4.3. Results and discussion

Table 1: Comparison between Newton-MOWAM and Newton-WAM solutions of Problem 1

m	Iterative method	ℓ_{max}	s	Maximum error
64	Newton-WAM	10605	5.09	8.508×10^{-05}
	Newton-MOWAM	9208	4.74	8.508×10^{-05}
128	Newton-WAM	36859	35.57	8.440×10^{-05}
	Newton-MOWAM	31123	33.09	8.440×10^{-05}
256	Newton-WAM	126489	254.59	8.163×10^{-05}
	Newton-MOWAM	105046	238.01	8.161×10^{-05}
512	Newton-WAM	428365	1788.66	7.046×10^{-05}
	Newton-MOWAM	358022	1663.95	7.038×10^{-05}
1024	Newton-WAM	1339097	12668.48	3.467×10^{-05}
	Newton-MOWAM	1116806	11794.36	3.388×10^{-05}

Table 2: Comparison between Newton-MOWAM and Newton-WAM solutions of Problem 2

m	Iterative method	ℓ_{max}	s	Maximum error
64	Newton-WAM	113	0.05	5.429×10^{-06}
	Newton-MOWAM	89	0.04	5.429×10^{-06}
128	Newton-WAM	342	0.22	3.870×10^{-06}
	Newton-MOWAM	172	0.12	3.865×10^{-06}
256	Newton-WAM	1165	1.25	3.478×10^{-06}
	Newton-MOWAM	337	0.40	3.466×10^{-06}
512	Newton-WAM	4117	8.45	3.380×10^{-06}
	Newton-MWAM	661	1.55	3.339×10^{-06}
1024	Newton-WAM	14584	59.01	3.355×10^{-06}
	Newton-MOWAM	1310	5.60	3.225×10^{-06}

Table 3: Comparison between Newton-MOWAM and Newton-WAM solutions of Problem 3

$m \times m$	Iterative method	ℓ_{max}	s	Maximum error
16×16	Newton-WAM	875	1.04	2.8248×10^{-03}
	Newton-MOWAM	243	0.52	2.8248×10^{-03}
32×32	Newton-WAM	3145	13.11	2.8455×10^{-03}
	Newton-MOWAM	483	3.32	2.8455×10^{-03}
64×64	Newton-WAM	11267	188.94	2.8473×10^{-03}
	Newton-MOWAM	939	23.05	2.8472×10^{-03}
128×128	Newton-WAM	40013	2934.48	2.8479×10^{-03}
	Newton-MWAM	1847	168.96	2.8478×10^{-03}
256×256	Newton-WAM	140551	56148.36	2.8482×10^{-03}
	Newton-MOWAM	3843	2043.62	2.8477×10^{-03}

Table 4: Percentage of reduction in the number of iterations and the computation time by Newton-MOWAM

Problem 1	m	Reduction in ℓ_{max} (%)	Reduction in s (%)
	64	13.17	6.88
	128	15.56	6.97
	256	16.95	6.51
	512	16.42	6.97
	1024	16.60	6.90
Problem 2	m	Reduction in ℓ_{max} (%)	Reduction in s (%)
	64	21.24	20.00
	128	49.71	45.45
	256	71.07	68.00
	512	83.94	81.66
	1024	91.02	90.51
Problem 3	$m \times m$	Reduction in ℓ_{max} (%)	Reduction in s (%)
	16×16	72.23	50.00
	32×32	84.64	74.68
	64×64	91.67	87.80
	128×128	95.38	94.24
	256×256	97.27	96.36

The numerical results from the simulation of Newton-MOWAM and Newton-WAM methods to solve Problem ~~1 and 2~~ 1, 2 and 3 are recorded accordingly from the C program outputs. The collected results such as the number of iterations, the computation time based on the time of completion by the developed C programs and the values of absolute errors are tabulated in Table ~~1 and 2~~ 1, 2 and 3. Besides that, the percentage of reduction in the number of iterations and the computation time is calculated and showed in Table ~~3~~ 4. Based on Table ~~1 and 2~~ 1, 2 and 3, which correspond to the results of Problem ~~1 and 2~~ 1, 2 and 3 respectively, the study found that the number of iterations and the simulation execute time required by Newton-MOWAM are notably lower than Newton-WAM for all five different ~~mesh sizes~~ orders of a matrix. The use of two optimum weighted parameters, which have been added to both forward and backward iterations, successfully improved the rate of convergence of the solutions of ~~Problem 1 and 2~~ tested problems. Moreover, the Newton-MOWAM method has minimized the number of iterations tremendously by 15.74% and the computation time by 6.85% for Problem 1 ~~and~~ , cut down the number of iterations by 63.40% while the computation time by 61.12% for Problem 2~~, see Table 3~~ and significantly reduced the number of iterations by 88.24% and the computation time by 80.62% for Problem 3, see Table 4.

In terms of numerical accuracy, the two tested iterative methods are comparable for the selected ~~mesh sizes~~ orders of a matrix. The study found that the magnitude of maximum absolute errors by Newton-MOWAM are smaller than by Newton-WAM, which can be seen at $m = 512, 1024$ for Problem 1 ~~and~~ , at $m = 128, 256, 512, 1024$ for Problem 2 and at $m = 64 \times 64, 128 \times 128, 256 \times 256$ for Problem 3. Evidently can be observed that the magnitude of maximum absolute errors by Newton-MOWAM is decreased further when the ~~mesh sizes~~ orders of a matrix is increased. In the other words, the approximate solution by Newton-MOWAM is converged to the exact solution at a considerably large ~~mesh~~ matrix size. Overall, these findings demonstrated that the efficacy of the proposed Newton-MOWAM method to obtain accurate solutions of nonlinear porous medium type equation with minimal computational complexity.

5. Conclusions

This paper presented the formulation of a new numerical method called Newton-MOWAM for solving several porous medium type equations. The analysis of efficiency and convergence of the method have also been discussed and supported by the results extracted from the C language based simulation. The numerical results of all the tested examples showed that the proposed Newton-MOWAM iterative method require~~s~~d least number of iterations with minimum execution time compared to the typical Newton-WAM method. Further, the Newton-MOWAM is also helped to obtain the results, which extremely closer to the exact solution when the ~~mesh~~ the order of matrix is large. Overall, it can be concluded that the Newton-MOWAM iterative method is certainly better approach in solving nonlinear porous medium type equation after the well-presented symmetry of the finite difference approximation and MOWAM iteration. In the future works, the applicability and efficiency of the Newton-MOWAM method will be further investigated to solve fractional porous medium type equations. Besides that, the Newton-MOWAM method is also will be extended to solve more complex equations such as thermo-elastic–viscoplastic medium with pores and thermo elastic medium with double porosity.

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