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Solving One-Dimensional Porous Medium Equation Using Unconditionally Stable Half-Sweep Finite Difference and SOR Method

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Abstract A porous medium equation is a nonlinear parabolic partial differential equation that presents many physical occurrences. The solutions of the porous medium equation are important to facilitate the investigation on nonlinear processes involving fluid flow, heat transfer, diffusion of gas-particles or population dynamics. As part of the development of a family of efficient iterative methods to solve the porous medium equation, the Half-Sweep technique has been adopted. Prior works in the existing literature on the application of Half-Sweep to successfully approximate the solutions of several types of mathematical problems are the underlying motivation of this research. This work aims to solve the one-dimensional porous medium equation efficiently by incorporating the Half-Sweep technique in the formulation of an unconditionally-stable implicit finite difference scheme. The noticeable unique property of Half-Sweep is its ability to secure a low computational complexity in computing numerical solutions. This work involves the application of the Half-Sweep finite difference scheme on the general porous medium equation, until the formulation of a nonlinear approximation function. The Newton method is used to linearize the formulated Half-Sweep finite difference approximation, so that the linear system in the form of a matrix can be constructed. Next, the Successive Over Relaxation method with a single parameter was applied to efficiently solve the generated linear system per time step. Next, to evaluate the efficiency of the developed method, deemed as the Half-Sweep Newton Successive Over Relaxation (HSNSOR) method, the criteria such as the number of iterations, the program execution time and the magnitude of absolute errors were investigated. According to the numerical results, the numerical solutions obtained by the HSNSOR are as accurate as those of the Half-Sweep Newton Gauss-Seidel (HSNGS), which is under the same family of Half-Sweep iterations, and the benchmark, Newton-Gauss-Seidel (NGS) method. The improvement in the numerical results produced by the HSNSOR is significant, and requires a lesser number of iterations and a shorter program execution time, as compared to the HSNGS and NGS methods.

Keywords One-Dimensional Porous Medium Equation, Half-Sweep, Finite Difference Method, Newton, Successive Over Relaxation, Iterative Method

1. Introduction

Porous medium equation (PME) is a nonlinear parabolic partial differential equation that exists in many nonlinear physical occurrences. For instance, PME is a general equation that brings up the Boussinesq equation that is used to model groundwater flow. PME is also used to describe the flow of ideal gas in a homogeneous porous medium, which is formulated by the laws such as mass balance, Darcy's law and state equation. In addition, PME is an important equation to be solved for a better understanding of the theory of heat propagation, particularly involving temperature-dependent thermal conductivity [1].

From the application of the PME side, [2] analyzed the heat transfer through human tissue, and found that the transport theory of porous media can be applied onto the biological heat transfer, as the theory reduces the number of assumptions when compared to other existing biological heat models. Then, [3] studied the qualitative properties of the PME in order to describe the dispersal processes in the dynamics of living things. The author found that the PME can be used to improve the qualitative as well as the quantitative agreement of population dynamics models. PME, without doubt, has great importance in many scientific fields, and more details about the theory and application of PME are available in [1].

The solutions of several one-dimensional PME problems via the finite difference method have been studied by many researchers [4-9]. As part of the development of a family of efficient iterative methods to solve the PME, this research adopted the Half-Sweep technique in the formulation of the finite difference method. Several researchers have discussed the success of the Half-Sweep technique in approximating the solutions of several types of mathematical problems [10-16]. Motivated by the unique property of Half-Sweep in securing a low computational complexity while computing the numerical solutions, this work aims to solve the one-dimensional PME using the unconditionally stable Half-Sweep finite difference approximation.

For this particular nonlinear type of partial difference equation, the finite difference discretization through the implementation of Half-Sweep yields a nonlinear type of approximation equation. Before the solution of PME is computed, the formulated nonlinear approximation equation is linearized using the Newton method to form a sparse and large linear system. A Successive Over Relaxation (SOR) iterative method with optimum parameters was applied for an efficient solution to a generated linear system.

2. Half-Sweep Finite Difference Method

Let us consider the general form of the one-dimensional PME [17]:

$$\frac{\partial u}{\partial t} = \rho \frac{\partial}{\partial x}\left(u^m \frac{\partial u}{\partial x}\right), u(x,t), 0 \leq x, t \leq 1, \quad (1)$$

where ρ and m are assumed to be any rational numbers. It is worthy to mention that Eq. (1) can exist for all $x \in \mathbb{R}$ and $0 < t < \infty$. For our numerical study, we attempt to investigate the numerical solution of Eq. (1) in a rectangular domain, subject to the boundary and initial conditions, as follows:

$$u(0,t) = g_0(t), u(1,t) = g_1(t), u(x,0) = u_0(x), \quad (2)$$

where $g_0(t), g_1(t)$and $u_0(x)$are the prescribed functions based on the provided exact solutions.

Before we show the formulation of Half-Sweep finite difference approximation to Eq. (1), it is best to discuss the formulation of the standard implicit finite difference approximation to Eq. (1), since our proposed method is based on the implicit finite difference method. Now, by defining the approximate solutions to Eq. (1), $U_{p,j} = U(p\Delta x, j\Delta t)$, $p = 0,1,2,\ldots, M-1, j = 0,1,2,\ldots T$ and both spatial and temporal steps are $\Delta x = 1/M$ and $\Delta t = 1/T$ respectively, the standard implicit finite difference approximation equation to Eq. (1) can be written as follows [4, 7, 8, 18]:

$$\begin{aligned} &U_{p,j+1} - \alpha U_{p,j+1}^m U_{p+1,j+1} + 2\alpha U_{p,j+1}^{m+1} - \alpha U_{p,j+1}^m U_{p-1,j+1} \\ &-\beta m U_{p,j+1}^{m-1} U_{p+1,j+1}^2 + 2\beta m U_{p,j+1}^{m-1} U_{p+1,j+1} U_{p-1,j+1} \\ &-\beta m U_{p,j+1}^{m-1} U_{p-1,j+1}^2 = U_{p,j}, \end{aligned} \quad (3)$$

where $\alpha = \frac{\rho \Delta t}{\Delta x^2}$, $\beta = \frac{\alpha}{4}$, $p = 1,2,\ldots, M-1$ and $j = 0,1,2,\ldots, T$.

The approximation equation shown in Eq. (3) can also be known as the Full-Sweep finite difference approximation equation, because it approximates all mesh points in a bounded domain. Hence, Eq. (3) can be extended to develop our Half-Sweep finite difference approximation equation by lengthening the distance between two consecutive mesh points from Δx to $2\Delta x$, as follows [8]:

$$\begin{aligned} &U_{p,j+1} - \alpha U_{p,j+1}^m U_{p+2,j+1} + 2\alpha U_{p,j+1}^{m+1} - \alpha U_{p,j+1}^m U_{p-2,j+1} \\ &-\beta m U_{p,j+1}^{m-1} U_{p+2,j+1}^2 + 2\beta m U_{p,j+1}^{m-1} U_{p+2,j+1} U_{p-2,j+1} \\ &-\beta m U_{p,j+1}^{m-1} U_{p-2,j+1}^2 = U_{p,j}, \end{aligned} \quad (4)$$

where $\alpha = \frac{\rho \Delta t}{4\Delta x^2}$, $\beta = \frac{\alpha}{4}$, $p = 2,4,\ldots, M-2$ and $j = 0,1,2,\ldots, T$.

The approximation equation (4) is proven to be unconditionally stable, and the proof is at the appendix.

Using Eq. (4), we may obtain a nonlinear system for time level $j+1$, in the form of:

$$F_{j+1} = 0, \quad (5)$$

where $F_{j+1} = \left(f_{2,j+1}, f_{4,j+1}, \ldots, f_{M-2,j+1}\right)^T$ and for each function,

$$\begin{aligned} &U_{p,j+1} - \alpha U_{p,j+1}^m U_{p+2,j+1} + 2\alpha U_{p,j+1}^{m+1} - \alpha U_{p,j+1}^m U_{p-2,j+1} \\ &-\beta m U_{p,j+1}^{m-1} U_{p+2,j+1}^2 + 2\beta m U_{p,j+1}^{m-1} U_{p+2,j+1} U_{p-2,j+1} \\ &-\beta m U_{p,j+1}^{m-1} U_{p-2,j+1}^2 - U_{p,j} = f_{p,j+1}, \end{aligned} \quad (6)$$

Since solving the nonlinear system (5) deals with great computational cost, we use the Newton method to linearize the nonlinear system (5), and then apply the SOR iterative method to obtain the solution. Using the Newton method, the linear system can be written as follows [7, 8, 19]:

$$A_{j+1}^{(k)} \underline{U'}_{j+1}^{(k)} = -F_{j+1}^{(k)}, \tag{7}$$

where

$$A_{j+1}^{(k)} = \begin{bmatrix} \frac{\partial f_{2,j+1}}{\partial U_{2,j+1}} & \frac{\partial f_{2,j+1}}{\partial U_{4,j+1}} & \cdots & \frac{\partial f_{2,j+1}}{\partial U_{M-2,j+1}} \\ \frac{\partial f_{4,j+1}}{\partial U_{2,j+1}} & \frac{\partial f_{4,j+1}}{\partial U_{4,j+1}} & \cdots & \frac{\partial f_{4,j+1}}{\partial U_{M-2,j+1}} \\ \vdots & \vdots & & \vdots \\ \frac{\partial f_{M-2,j+1}}{\partial U_{2,j+1}} & \frac{\partial f_{M-2,j+1}}{\partial U_{4,j+1}} & \cdots & \frac{\partial f_{M-2,j+1}}{\partial U_{M-2,j+1}} \end{bmatrix}^{(k)},$$

$$\underline{U'}_{j+1}^{(k)} = \begin{bmatrix} U'_{2,j+1} \\ U'_{4,j+1} \\ \vdots \\ U'_{M-2,j+1} \end{bmatrix}^{(k)}, \text{ and } F_{j+1}^{(k)} = \begin{bmatrix} f_{2,j+1} \\ f_{4,j+1} \\ \vdots \\ f_{M-2,j+1} \end{bmatrix}^{(k)}.$$

The approximate solutions to Eq. (1) are computed by:

$$\underline{U}_{j+1}^{(k+1)} = \underline{U}_{j+1}^{(k)} + \underline{U'}_{j+1}^{(k)}, \tag{8}$$

where $\underline{U}_{j+1}^{(k)} = \left(U_{2,j+1}^{(k)}, U_{4,j+1}^{(k)}, \dots, U_{M-2,j+1}^{(k)}\right)^T$.

3. HSNSOR Iterative Method

Based on the linear system (7), we find out that the coefficient matrix $A_{j+1}^{(k)}$ has the form of a tridiagonal. Thus, to apply the SOR iterative method for solving the linear system (7) [20, 21], we consider the three components' decomposition of $A_{j+1}^{(k)}$ as follows,

$$A_{j+1}^{(k)} = D_{j+1}^{(k)} - L_{j+1}^{(k)} - V_{j+1}^{(k)}, \tag{9}$$

where $D_{j+1}^{(k)}$ is the diagonal of the matrix, $L_{j+1}^{(k)}$ is the strictly lower triangular matrix, and $V_{j+1}^{(k)}$ is the strictly upper triangular matrix, at the time level $j+1$ and k-th iteration.

Hence, using the linear system (7) and the decomposition (9), the proposed method (HSNSOR) can be derived into

$$\underline{U'}_{j+1}^{(k+1)} = (1-\omega)\underline{U'}_{j+1}^{(k)} + \omega\left(D_{j+1}^{(k)} - L_{j+1}^{(k)}\right)^{-1}\left(V_{j+1}^{(k)}\underline{U'}_{j+1}^{(k)} - F_{j+1}^{(k)}\right), \tag{10}$$

Based on the formula shown in (10), the relaxation parameter lies within $1 < \omega < 2$. When $\omega = 1$, the formula can be known as the HSNGS [8].

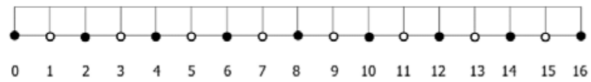

Figure 1. HSNSOR on the finite mesh points $M = 16$

According to Figure 1, the implementation of the HSNSOR method for solving Eq. (1) can be explained as follows. After the boundary and initial conditions were imposed on the solution domain, the HSNSOR approximates the solutions on all the interior mesh points that are labelled by black dots, i.e. 2, 4, …, 14. After the iteration process is completed and the values of the black dots are obtained, the remaining mesh points that are labelled by white dots, i.e. 1, 3, …, 15 are computed directly using the approximation equation. The full algorithm for the computation using the HSNSOR method is described in Algorithm 1.

Algorithm 1. HSNSOR iterative method

i. At time level j, define $g_0(t), g_1(t)$ and $u_0(x)$,
ii. Initialize the value of ω, $\underline{U}_{j+1}^{(k)} = 1.0$, and $\underline{U'}_{j+1}^{(k)} = 0$,
iii. Set up the linear system (7),
iv. Iterate the formula (10),
v. Check the convergence $\left|\underline{U'}_{j+1}^{(k+1)} - \underline{U'}_{j+1}^{(k)}\right| < 10^{-10}$. If the correctors converge, compute (8) and then the remaining mesh points,
vi. Check the convergence for all mesh points using $\left|F_{j+1}^{(k+1)} - F_{j+1}^{(k)}\right| < 10^{-10}$. If the solutions converge, go to $j+1$.

In practice, the optimum value of ω is determined (±0.01) by running Algorithm 1 several times, and the one that gives the least number of iterations is selected as the optimum.

4. Stability Analysis of the Half-Sweep Finite Difference Method on the One-Dimensional Porous Medium Equation

The application of Fourier analysis to prove the stability of the applied finite-difference on nonlinear partial differential equation (like PME) cannot be rigorously justified. Nevertheless, it is practically effective [22].

Assuming the solution $u(x,t)$ exists within the region of $0 < x, t \leq 1$. Additionally, "freeze" the nonlinear term u^m at each mesh point in the same region, and let it be a constant μ. Eq. (1) can be rewritten into the following:

$$\frac{\partial u}{\partial t} = \rho \frac{\partial}{\partial x}\left(\mu \frac{\partial u}{\partial x}\right) = \rho\mu \frac{\partial^2 u}{\partial x^2}. \tag{11}$$

The Half-Sweep finite difference method that is used to discretize Eq. (11) can be defined as:

$$\frac{\partial u}{\partial t} = D_{-t}U_{p,j+1} = \frac{U_{p,j+1} - U_{p,j}}{\Delta t}, \tag{12}$$

and

$$\frac{\partial^2 u}{\partial x^2} = D_{+2x}D_{-2x}U_{p,j+1} = \frac{U_{p+2,j+1} - 2U_{p,j+1} + U_{p-2,j+1}}{4(\Delta x)^2}. \tag{13}$$

Using Eq. (12) and (13) to discretize Eq. (11) gives:

$$U_{p,j+1} - \lambda\left(U_{p+2,j+1} - 2U_{p,j+1} + U_{p-2,j+1}\right) = U_{p,j}, \tag{14}$$

where $\lambda = \rho\mu(\Delta t)/4(\Delta x)^2$.

By applying the von Neumann method which is:

$$U_{p,j} = \xi^j e^{p\theta i}, \tag{15}$$

Eq. (14) can be transformed into:

$$\xi\left(1 - \lambda\left(e^{2\theta i} - 2 + e^{-2\theta i}\right)\right) = 1. \tag{16}$$

Since $e^{2\theta i} - 2 + e^{-2\theta i} = -4\sin^2\theta$, Eq. (16) can be further rewritten into:

$$\xi = \frac{1}{1 + 4\lambda \sin^2\theta}. \tag{17}$$

Based on Eq. (17), we have $0 < \xi \leq 1$ for all positive values of λ and $\theta \in [-\pi, \pi]$. Hence, the Half-Sweep finite difference approximation is proven to be unconditionally stable.

5. Numerical Experiment

For the numerical experiment, several criteria are observed such as the number of iterations (k), the program execution time (seconds) and the magnitude of absolute errors (ε_{max}). These criteria are used to evaluate the efficiency of the HSNSOR method to solve Eq. (1) subjects to both initial and boundary conditions as in Eq. (2). The efficiency of the HSNSOR method is then compared to the HSNGS and NGS [23] methods using four selected examples. Four examples used for the numerical experiment are presented hereafter.

Example 1 [17]

Given a one-dimensional PME with m equals to 1:

$$\frac{\partial u}{\partial t} = \rho\frac{\partial}{\partial x}\left(u\frac{\partial u}{\partial x}\right), \tag{18}$$

and the exact solution $u(x,t) = C_1x + C_1^2t + C_2$ with the arbitrary constants C_1 and C_2. This experiment uses $C_1 = 1$ and $C_2 = 0$.

Example 2 [17]

Given a one-dimensional PME wherein m equals to -1 and the parameter ρ is 0.5:

$$\frac{\partial u}{\partial t} = 0.5\frac{\partial}{\partial x}\left(u^{-1}\frac{\partial u}{\partial x}\right). \tag{19}$$

The exact solution is $u(x,t) = (C_1x - 0.5C_1^2t + C_2)^{-1}$, and this experiment uses $C_1 = 0.6$ and $C_2 = 1.3$.

Example 3 [24]

Given a one-dimensional PME wherein m equals to 2:

$$\frac{\partial u}{\partial t} = \rho\frac{\partial}{\partial x}\left(u^2\frac{\partial u}{\partial x}\right), \tag{20}$$

and the exact solution$u(x,t) = (x+1)\left(2\sqrt{C^2 - t}\right)^{-1}$ has a condition $t < C^2$. For the experiment, we use $C = 2$.

Example 4 [24]

Given a one-dimensional PME wherein m equals to -2 and the parameter ρ is 0.5:

$$\frac{\partial u}{\partial t} = 0.5\frac{\partial}{\partial x}\left(u^{-2}\frac{\partial u}{\partial x}\right). \tag{21}$$

The exact solution is $u(x,t) = (2C_1x - c_1^2t + C_2)^{-\frac{1}{2}}$ and we use $C_1 = 0.35$ and $C_2 = 1.35$.

Numerical outputs collected from the implementation of the HSNSOR, HSNGS and NGS methods on the four examples are tabulated in Tables 1 to 4. The tables show the comparison between the three implemented methods based on the number of iterations (k), the program execution time (seconds) and the magnitude of absolute errors(ε_{max}) with five different sizes of mesh points, M. Additionally, Table 5 shows the percentages of reduction in the number of iterations and the program execution time by the HSNSOR and HSNGS against the control method, NGS.

Table 1. The numerical results of Example 1

M	Method (ω)	k	seconds	ε_{max}
64	NGS	3835	2.38	2.76×10^{-8}
	HSNGS	1065	0.16	6.16×10^{-9}
	HSNSOR (1.59)	269	0.14	1.84×10^{-10}
128	NGS	13678	7.50	1.22×10^{-7}
	HSNGS	3835	0.86	2.75×10^{-8}
	HSNSOR (1.77)	562	0.32	1.19×10^{-10}
256	NGS	48395	38.58	5.33×10^{-7}
	HSNGS	13678	5.62	1.22×10^{-7}
	HSNSOR (1.87)	1142	1.13	2.09×10^{-10}
512	NGS	169693	252.94	2.10×10^{-6}
	HSNGS	48395	38.22	5.33×10^{-7}
	HSNSOR (1.93)	2328	3.68	3.19×10^{-10}
1024	NGS	587031	1712.49	7.62×10^{-6}
	HSNGS	169693	274.28	2.10×10^{-6}
	HSNSOR (1.97)	4942	17.25	9.10×10^{-11}

Table 2. The numerical results of Example 2

M	Method (ω)	k	seconds	ε_{max}
64	NGS	1720	1.13	2.03×10^{-5}
	HSNGS	489	0.20	2.03×10^{-5}
	HSNSOR (1.48)	186	0.15	2.03×10^{-5}
128	NGS	6034	4.06	2.02×10^{-5}
	HSNGS	1720	1.07	2.03×10^{-5}
	HSNSOR (1.69)	375	0.35	2.03×10^{-5}
256	NGS	20907	27.03	2.00×10^{-5}
	HSNGS	6034	6.45	2.02×10^{-5}
	HSNSOR (1.83)	745	1.16	2.03×10^{-5}
512	NGS	71385	287.34	1.93×10^{-5}
	HSNGS	20907	43.75	2.00×10^{-5}
	HSNSOR (1.91)	1464	3.78	2.03×10^{-5}
1024	NGS	239975	1741.01	1.72×10^{-5}
	HSNGS	71385	304.92	1.93×10^{-5}
	HSNSOR (1.95)	3044	13.95	2.03×10^{-5}

Table 3. The numerical results of Example 3

M	Method (ω)	k	seconds	ε_{max}
64	NGS	1344	1.17	8.39×10^{-5}
	HSNGS	386	0.17	8.38×10^{-5}
	HSNSOR (1.52)	231	0.15	8.38×10^{-5}
128	NGS	4824	2.84	8.39×10^{-5}
	HSNGS	1344	0.75	8.39×10^{-5}
	HSNSOR (1.73)	461	0.38	8.39×10^{-5}
256	NGS	17308	20.03	8.39×10^{-5}
	HSNGS	4824	4.71	8.39×10^{-5}
	HSNSOR (1.85)	908	1.25	8.39×10^{-5}
512	NGS	61658	270.11	8.40×10^{-5}
	HSNGS	17308	33.05	8.39×10^{-5}
	HSNSOR (1.92)	1784	4.25	8.39×10^{-5}
1024	NGS	218147	2008.35	8.43×10^{-5}
	HSNGS	61658	227.65	8.40×10^{-5}
	HSNSOR (1.96)	3490	15.77	8.39×10^{-5}

Table 4. The numerical results of Example 4

M	Method (ω)	k	seconds	ε_{max}
64	NGS	2015	1.26	2.88×10^{-6}
	HSNGS	562	0.23	2.65×10^{-6}
	HSNSOR (1.50)	205	0.16	2.66×10^{-6}
128	NGS	7082	4.90	2.90×10^{-6}
	HSNGS	2015	1.23	2.88×10^{-6}
	HSNSOR (1.70)	420	0.36	2.90×10^{-6}
256	NGS	24325	45.42	2.71×10^{-6}
	HSNGS	7082	7.37	2.90×10^{-6}
	HSNSOR (1.84)	837	1.29	2.96×10^{-6}
512	NGS	81729	354.79	1.86×10^{-6}
	HSNGS	24325	50.33	2.71×10^{-6}
	HSNSOR (1.92)	1706	4.18	2.97×10^{-6}
1024	NGS	265698	2293.23	3.33×10^{-6}
	HSNGS	81729	332.37	1.86×10^{-6}
	HSNSOR (1.96)	3381	14.79	2.98×10^{-6}

Table 5. Percentages of reduction in the number of iterations and the program execution time by HSNSOR and HSNGS

Example	Iterative Method	k (%)	seconds (%)
1	HSNSOR	92.99-99.16	94.12-98.99
	HSNGS	71.09-72.23	83.98-93.28
2	HSNSOR	89.19-98.73	86.73-99.20
	HSNGS	70.25-71.57	73.65-84.77
3	HSNSOR	82.81-98.40	86.62-99.21
	HSNGS	71.28-72.14	73.59-88.66
4	HSNSOR	89.83-98.73	87.30-99.36
	HSNGS	69.24-72.11	74.90-85.81

6. Conclusion

In conclusion, we have successfully derived and implemented the HSNSOR method for solving linearized systems formed by considering several mesh points and the Half-Sweep implicit finite difference approximation equation. According to the numerical results, the HSNSOR method has successfully reduced the number of iterations by approximately 82.81-99.16%, and the program execution time by approximately 86.62-99.36% in solving the one-dimensional PME, when compared to the NGS method (Table 5). This significant improvement is attributed to the usage of the optimum values of ω for the SOR iterative method. Another reason is that the application of Half-Sweep contributes to the reduction of computational complexity. Overall, all methods have a good agreement in terms of accuracy.

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